University of Ottawa

Department of Mathematics and Statistics

MAT 2125: Elementary Real Analysis
Professor: Alistair Savage

Final Exam
April 2017

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	11	Total
Maximum	5	4	4	3	4	4	3	5	4	4	4	44
Grade												

QUESTION 1. **[5 points]** For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers. In all of the statements, a and b are real numbers with $a < b$.

Grading: You will receive 0.5 points for each correct answer. You will not lose points for incorrect answers.

____ If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions converging pointwise to a function f, then f is continuous.

____ Every continuous function $f\colon [-\pi, \pi] \to \mathbb{R}$ has a Fourier series.

____ Every differentiable function $f\colon [a, b] \to \mathbb{R}$ is integrable on $[a, b]$.

____ If $f\colon [a, b] \to \mathbb{R}$ has a local extremum at $x_0 \in [a, b]$, then $f'(x_0) = 0$.

____ If $f\colon [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and $f'(x_0) = 0$ for some $x_0 \in [a, b]$, then f has a local extremum at x_0.

____ If $f\colon [a, b] \to \mathbb{R}$ is continuous and injective, then f is either strictly increasing or strictly decreasing.

____ If a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}^d$ converges to $L \in \mathbb{R}^d$, then every subsequence of $\{a_n\}_{n=1}^{\infty}$ also converges to L.

____ Every Cauchy sequence of real numbers is monotonic.

____ Every function represented by a power series is continuous on its interval of convergence.

____ Every continuous function is equal to its Taylor series on the interval of convergence of the series.

QUESTION 2. Consider the sequence $\{a_n\}_{n=1}^{\infty}$, with

$$a_n = \begin{cases} -n & \text{if } n \text{ is divisible by 3,} \\ 2 - \frac{1}{n} & \text{if } n \text{ is not divisible by 3.} \end{cases}$$

(a) [**3 points**] Find $\limsup_{n\to\infty} a_n$.

(b) [**1 point**] Does $\lim_{n\to\infty} a_n$ exist? Remember to justify your answer.

QUESTION 3. [**4 points**] Is the series

$$\sum_{k=2}^{\infty} \frac{(-1)^k (k^2 - k - 1)}{k^4 + 5}$$

convergent? Is it absolutely convergent? *Note*: Answer both questions with a clear 'yes' or 'no', in addition to your justification.

QUESTION 4. [**3 points**] Consider the step function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Compute the one-sided derivatives $f'_-(0)$ and $f'_+(0)$. Is f differentiable at 0?

QUESTION 5.

(a) [**1 point**] State the Heine–Borel Theorem for $\mathbb{R}^d$.

(b) [**2 points**] Prove that a union of two compact subsets of $\mathbb{R}^d$ is compact. *Note*: There are many ways to prove this. You may use the definition of compact sets in terms of open covers, the equivalent notion of sequential compactness, or the Heine–Borel Theorem.

(c) **[1 point]** Give an example of compact sets K_i, $i \in \mathbb{N}$, such that $\bigcup_{i=1}^{\infty} K_i$ is not compact. Justify your answer (i.e., justify that your example has the required properties).

QUESTION 6.

(a) [**1 point**] Given the definition of *uniform continuity.* More precisely, state what it means for a function $f\colon A \to \mathbb{R}^m$ (where $A \subseteq \mathbb{R}^d$) to be uniformly continuous on A.

(b) [**3 points**] Is the function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2,$$

uniformly continuous on $\mathbb{R}$? Remember to justify your answer.

QUESTION 7.

(a) [**1 point**] State the Mean Value Theorem (for derivatives) as stated in class.

(b) [**2 points**] Suppose the equation

$$x^5 + \alpha x - 10 = 0$$

has at least two solutions. Prove that $\alpha \leq 0$.

QUESTION 8. Define $f\colon [-1,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } -1 \le x < 0, \\ 0 & \text{if } x = 0, \\ -2 & \text{if } 0 < x \le 1. \end{cases}$$

For $n \in \mathbb{N}$, $n \ge 2$, define the partition

$$P_n = \left\{ -1, \frac{-1}{n}, \frac{1}{n}, 1 \right\}.$$

of $[-1,1]$.

(a) **[3 points]** Compute the upper sum $U(P_n, f)$ and lower sum $L(P_n, f)$.

(b) **[2 points]** Prove that f is integrable on $[-1, 1]$ and find $\int_{-1}^{1} f$.

QUESTION 9. [**4 points**] For which values of $s \in \mathbb{N}$ is the integral

$$\int_2^\infty \frac{1}{x^s}\, dx$$

convergent? For the values of s for which it converges, compute the integral.

QUESTION 10. Let $A \subseteq \mathbb{R}^d$. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions with $f_n \colon A \to \mathbb{R}^m$ for each $n \in \mathbb{N}$. Furthermore, suppose $f \colon A \to \mathbb{R}^m$.

(a) [**1 point**] Define pointwise convergence. More precisely, state what it means for the sequence $\{f_n\}_{n=1}^{\infty}$ to *converge pointwise* to f on A. (You may use limits of sequences of real numbers in your answer.)

(b) [**1 point**] Define uniform convergence. More precisely, state what it means for the sequence $\{f_n\}_{n=1}^{\infty}$ to *converge uniformly* to f on A.

(c) [**2 points**] Fix $a \in \mathbb{R}$ such that $0 < a < 1$. Consider the sequence of functions $\{g_n\}_{n=1}^{\infty}$ defined by

$$g_n \colon [-a, a] \to \mathbb{R}, \quad g_n(x) = x^n \sin(3x),$$

for $n \in \mathbb{N}$. Prove that the series $\sum_{n=1}^{\infty} g_n$ converges uniformly on $[-a, a]$. Remember to cite any theorems that you use.

QUESTION 11. Fix $a \in (0, \infty)$. Consider the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k a^k}.$$

(a) [**1 point**] What is the radius of convergence of this power series?

(b) [**2 points**] Find the interval of convergence of the series.

(c) [**1 point**] What is $f^{(25)}(0)$?

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