University of Ottawa

Department of Mathematics and Statistics

MAT 2125: Elementary Real Analysis

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Final Exam – Solutions

April 2017

Surname ______________________ First Name ______________________

Student # ______________________

Instructions:

(a) You have 3 hours to complete this exam.

(b) The number of points available for each question is indicated in square brackets.

(c) Unless otherwise indicated, you must justify your answers to receive full marks.

(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(e) Write your student number at the top of each page in the space provided.

(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	11	Total
Maximum	5	4	4	3	4	4	3	5	4	4	4	44
Grade												

QUESTION 1. **[5 points]** For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers. In all of the statements, a and b are real numbers with $a < b$.

Grading: You will receive 0.5 points for each correct answer. You will not lose points for incorrect answers.

- F If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions converging pointwise to a function f, then f is continuous.
- T Every continuous function $f\colon [-\pi, \pi] \to \mathbb{R}$ has a Fourier series.
- T Every differentiable function $f\colon [a, b] \to \mathbb{R}$ is integrable on $[a, b]$.
- F If $f\colon [a, b] \to \mathbb{R}$ has a local extremum at $x_0 \in [a, b]$, then $f'(x_0) = 0$.
- F If $f\colon [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and $f'(x_0) = 0$ for some $x_0 \in [a, b]$, then f has a local extremum at x_0.
- T If $f\colon [a, b] \to \mathbb{R}$ is continuous and injective, then f is either strictly increasing or strictly decreasing.
- T If a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}^d$ converges to $L \in \mathbb{R}^d$, then every subsequence of $\{a_n\}_{n=1}^{\infty}$ also converges to L.
- F Every Cauchy sequence of real numbers is monotonic.
- T Every function represented by a power series is continuous on its interval of convergence.
- F Every continuous function is equal to its Taylor series on the interval of convergence of the series.

QUESTION 2. Consider the sequence $\{a_n\}_{n=1}^{\infty}$, with

$$a_n = \begin{cases} -n & \text{if } n \text{ is divisible by } 3, \\ 2 - \frac{1}{n} & \text{if } n \text{ is not divisible by } 3. \end{cases}$$

(a) [**3 points**] Find $\limsup_{n\to\infty} a_n$.

Solution: Note that $-n < 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\left(2 - \frac{1}{n}\right) = 2$. Thus, for $n \in \mathbb{N}$, we have

$$\sup\{a_n, a_{n+1}, a_{n+2}, \dots\} = 2.$$

Thus,

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+2}, \dots\} = 2.$$

(b) [**1 point**] Does $\lim_{n\to\infty} a_n$ exist? Remember to justify your answer.

Solution: No, it does not exist. Note that, for all $n \in \mathbb{N}$, we have

$$|a_{n+1} - a_n| \geq n.$$

So the sequence is not Cauchy, hence does not converge.

Alternate solution: Letting $n_k = 3k$ for $k \in \mathbb{N}$, we get the subsequence $\{a_{n_k}\}_{k=1}^{\infty} = \{-3k\}_{k=1}^{\infty}$, which diverges to $-\infty$. Thus the sequence $\{a_n\}_{n=1}^{\infty}$ cannot converge. (Any subsequence of a convergent subsequence converges.)

QUESTION 3. [**4 points**] Is the series

$$\sum_{k=2}^{\infty} \frac{(-1)^k (k^2 - k - 1)}{k^4 + 5}$$

convergent? Is it absolutely convergent? *Note*: Answer both questions with a clear 'yes' or 'no', in addition to your justification.

Solution: For $k \geq 2$, we have

$$\left| \frac{(-1)^k (k^2 - k - 1)}{k^4 + 5} \right| = \frac{k^2 - k - 1}{k^4 + 5} \leq \frac{k^2}{k^4} = \frac{1}{k^2}.$$

Since

$$\sum_{k=2}^{\infty} \frac{1}{k^2}$$

converges, the series

$$\sum_{k=2}^{\infty} \frac{(-1)^k (k^2 - k - 1)}{k^4 + 5}$$

converges absolutely by the comparison test. Because all absolutely convergent series are convergent, the series also converges.

QUESTION 4. [**3 points**] Consider the step function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Compute the one-sided derivatives $f'_-(0)$ and $f'_+(0)$. Is f differentiable at 0?

Solution: We have

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{0-0}{x} = 0,$$

and

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{1-0}{x} = \infty.$$

Since the two one-sided limits do not agree, the function is *not* differentiable at 0. (Of course, we also know that it is not differentiable at 0 since it is not continuous there.)

QUESTION 5.

(a) **[1 point]** State the Heine–Borel Theorem for $\mathbb{R}^d$.

Solution: A subset of $\mathbb{R}^d$ is compact if and only if it is closed and bounded.

(b) **[2 points]** Prove that a union of two compact subsets of $\mathbb{R}^d$ is compact. *Note*: There are many ways to prove this. You may use the definition of compact sets in terms of open covers, the equivalent notion of sequential compactness, or the Heine–Borel Theorem.

Solution: *Solution 1*: Suppose K_1 and K_2 are compact subsets of $\mathbb{R}^d$. Let $\mathcal{U}$ be an open cover of the union $K_1 \cup K_2$. Then, for each $i \in \{1, 2\}$, there is a finite subcover $\mathcal{U}_i$ of K_i. Then $\mathcal{U}_1 \cup \mathcal{U}_2$ is a finite subcover of $K_1 \cup K_2$.

Solution 2: Suppose K_1, K_2 are compact subsets of $\mathbb{R}^d$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $K_1 \cup K_2$. Then at least one of the K_i ($i \in \{1, 2\}$) has the property that an infinite number of elements of the sequence lies in K_i. Since K_i is compact, there is a subsequence converging to a point in $K_i \subseteq K_1 \cup K_2$. Thus $K_1 \cup K_2$ is compact.

Solution 3: Suppose K_1, K_2 are compact subsets of $\mathbb{R}^d$. Then, by the Heine–Borel Theorem, they are closed and bounded. Since they are bounded, for each $i \in \{1, 2\}$, there exists $M_i > 0$ such that

$$\|x\| \leq M_i \quad \forall\, x \in K_i.$$

Setting $M = \max\{M_1, M_2\}$, we then see that

$$\|x\| \leq M \quad \forall\, x \in K_1 \cup K_2.$$

So the union is bounded. It is also closed, since the union of a finite number of closed sets is closed. Thus it is compact by the Heine–Borel Theorem.

(c) **[1 point]** Give an example of compact sets K_i, $i \in \mathbb{N}$, such that $\bigcup_{i=1}^{\infty} K_i$ is not compact. Justify your answer (i.e., justify that your example has the required properties).

Solution: For $i \in \mathbb{N}$, let $K_i = [-i, i]$. Then each K_i is closed and bounded, hence compact by the Heine–Borel Theorem. However, $\bigcup_{i=1}^{\infty} K_i = \mathbb{R}$ is not bounded, hence not compact.

QUESTION 6.

(a) **[1 point]** Given the definition of *uniform continuity*. More precisely, state what it means for a function $f\colon A \to \mathbb{R}^m$ (where $A \subseteq \mathbb{R}^d$) to be uniformly continuous on A.

Solution: The function $f\colon A \to \mathbb{R}^m$ is uniformly continuous on A if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in A,\ \|x - y\| < \delta \implies \|f(x) - f(y)\| < \varepsilon.$$

(b) **[3 points]** Is the function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2,$$

uniformly continuous on $\mathbb{R}$? Remember to justify your answer.

Solution: No, f is not uniformly continuous on $\mathbb{R}$. Note that

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y| \cdot |x + y|.$$

Consider $\varepsilon = 1$ and let $\delta > 0$. Then for $x = 1/\delta$ and $y = 1/\delta + \delta/2$, we have

$$|x - y| = \frac{\delta}{2} < \delta,$$

but

$$|f(x) - f(y)| = \frac{\delta}{2}\left(\frac{2}{\delta} + \frac{\delta}{2}\right) = 1 + \frac{\delta^2}{4} > 1 = \varepsilon.$$

QUESTION 7.

(a) [**1 point**] State the Mean Value Theorem (for derivatives) as stated in class.

Solution: Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) [**2 points**] Suppose the equation

$$x^5 + \alpha x - 10 = 0$$

has at least two solutions. Prove that $\alpha \leq 0$.

Solution: Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^5 + \alpha x - 10$. Since f is a polynomial, it is continuous and differentiable. Suppose the given equation has at least two solutions. Then there exist $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) = f(b) = 0$. Then, by the Mean Value Theorem (on the interval $[a,b]$), there exists a $c \in (a,b)$ such that $f'(c) = 0$. Thus

$$0 = f'(c) = 5c^4 + \alpha.$$

Therefore $\alpha = -5c^4 \leq 0$.

QUESTION 8. Define $f\colon [-1,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x < 0, \\ 0 & \text{if } x = 0, \\ -2 & \text{if } 0 < x \leq 1. \end{cases}$$

For $n \in \mathbb{N}$, $n \geq 2$, define the partition

$$P_n = \left\{-1, \frac{-1}{n}, \frac{1}{n}, 1\right\}.$$

of $[-1,1]$.

(a) [**3 points**] Compute the upper sum $U(P_n, f)$ and lower sum $L(P_n, f)$.

Solution: We have

$$M_1(P_n,f) = 1, \quad M_2(P_n,f) = 1, \quad M_3(P_n,f) = -2,$$
$$m_1(P_n,f) = 1, \quad m_2(P_n,f) = -2, \quad m_3(P_n,f) = -2.$$

Thus,

$$\begin{aligned} U(P_n,f) &= 1 \cdot \left(\frac{-1}{n} - (-1)\right) + 1 \cdot \left(\frac{1}{n} - \frac{-1}{n}\right) + (-2) \cdot \left(1 - \frac{1}{n}\right) \\ &= 1 - \frac{1}{n} + \frac{2}{n} - 2 + \frac{2}{n} \\ &= \frac{3}{n} - 1, \end{aligned}$$

and

$$\begin{aligned} U(P_n,f) &= 1 \cdot \left(\frac{-1}{n} - (-1)\right) + (-2) \cdot \left(\frac{1}{n} - \frac{-1}{n}\right) + (-2) \cdot \left(1 - \frac{1}{n}\right) \\ &= 1 - \frac{1}{n} - \frac{4}{n} - 2 + \frac{2}{n} \\ &= \frac{-3}{n} - 1. \end{aligned}$$

(b) [**2 points**] Prove that f is integrable on $[-1,1]$ and find $\int_{-1}^{1} f$.

Solution: Let $\varepsilon > 0$. Choose an integer $n > \max\{1, \frac{6}{\varepsilon}\}$. Then

$$U(P_n,f) - L(P_n,f) = \frac{6}{n} < \varepsilon.$$

Hence f is integrable on $[-1,1]$. Furthermore, for all $n \in \mathbb{N}$, we have

$$\frac{-3}{n} - 1 = L(P_n,f) \leq \int_{-1}^{1} f \leq U(P_n,f) = \frac{-3}{n} - 1.$$

Since $\lim_{n\to\infty} 3/n = 0$, we have $\int_{-1}^{1} f = -1$.

QUESTION 9. [**4 points**] For which values of $s \in \mathbb{N}$ is the integral

$$\int_2^\infty \frac{1}{x^s}\,dx$$

convergent? For the values of s for which it converges, compute the integral.

Solution: First suppose $s \neq 1$. Then

$$\begin{aligned}\int_2^\infty \frac{1}{x^s}\,dx &= \lim_{b\to\infty}\int_2^b \frac{1}{x^s}\,dx \\ &= \lim_{b\to\infty} \frac{1}{1-s}\frac{1}{x^{s-1}}\bigg|_2^b \\ &= \frac{1}{1-s}\lim_{b\to\infty}\left(\frac{1}{b^{s-1}} - \frac{1}{2^{s-1}}\right) \\ &= \frac{1}{(s-1)2^{s-1}},\end{aligned}$$

since $\lim_{b\to\infty} b^{1-s} = 0$ for $s \geq 2$.

Now suppose $s = 1$. Then

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} (\log x)|_2^b \\ &= \lim_{b\to\infty}(\log b - \log 2) \\ &= \infty.\end{aligned}$$

Therefore, the integral converges if and only if $s > 1$, in which case it is equal to $\frac{1}{(s-1)2^{s-1}}$.

QUESTION 10. Let $A \subseteq \mathbb{R}^d$. Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of functions with $f_n \colon A \to \mathbb{R}^m$ for each $n \in \mathbb{N}$. Furthermore, suppose $f \colon A \to \mathbb{R}^m$.

(a) **[1 point]** Define pointwise convergence. More precisely, state what it means for the sequence $\{f_n\}_{n=1}^\infty$ to *converge pointwise* to f on A. (You may use limits of sequences of real numbers in your answer.)

Solution: The sequence $\{f_n\}_{n=1}^\infty$ *converges pointwise* to f on A if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all } x \in A.$$

(b) **[1 point]** Define uniform convergence. More precisely, state what it means for the sequence $\{f_n\}_{n=1}^\infty$ to *converge uniformly* to f on A.

Solution: The sequence $\{f_n\}_{n=1}^\infty$ *converges uniformly* to f on A if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|f_n(x) - f(x)\| < \varepsilon \quad \text{for all } x \in A,\ n \geq N.$$

(c) **[2 points]** Fix $a \in \mathbb{R}$ such that $0 < a < 1$. Consider the sequence of functions $\{g_n\}_{n=1}^\infty$ defined by

$$g_n \colon [-a, a] \to \mathbb{R}, \quad g_n(x) = x^n \sin(3x),$$

for $n \in \mathbb{N}$. Prove that the series $\sum_{n=1}^\infty g_n$ converges uniformly on $[-a, a]$. Remember to cite any theorems that you use.

Solution: For all $n \in \mathbb{N}$ and $x \in [-a, a]$, we have

$$|g_n(x)| = |x^n \sin(3x)| = |x^n| \cdot |\sin(3x)| \leq |x^n| \leq a^n.$$

Since $\sum_{n=1}^\infty a^n$ is a convergent geometric series (because $0 < a < 1$), the series $\sum_{n=1}^\infty g_n$ converges uniformly by the Weierstass M-test.

QUESTION 11. Fix $a \in (0, \infty)$. Consider the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k a^k}.$$

(a) **[1 point]** What is the radius of convergence of this power series?

Solution: We have

$$\limsup_{k\to\infty} \sqrt[k]{\left|\frac{(-1)^k}{k a^k}\right|} = \limsup_{k\to\infty} \frac{1}{a\sqrt[k]{k}} = \frac{1}{a}.$$

Thus the radius of convergence is a.

(b) **[2 points]** Find the interval of convergence of the series.

Solution: Since we have already found the radius of convergence, it remains to check the endpoints $x = \pm a$. When $x = a$, the series becomes

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}.$$

This is the alternating harmonic, which converges. On the other hand, when $x = -a$, the series becomes

$$\sum_{k=1}^{\infty} \frac{1}{k},$$

which is the divergent harmonic series. Therefore, the interval of convergence is

$$(-a, a].$$

(c) **[1 point]** What is $f^{(25)}(0)$?

Solution: We know that the k-th coefficient of the power series (i.e., the coefficient of x^k) is given by

$$\frac{f^{(k)}(0)}{k!}.$$

Therefore,

$$\frac{(-1)^{25}}{25a^{25}} = \frac{f^{(25)}(0)}{25!}.$$

Hence

$$f^{(25)}(0) = \frac{(-1)^{25} 25!}{25a^{25}} = -\frac{24!}{a^{25}}.$$