uOttawa University of Ottawa

Faculty of Science »
Department of mathematics and statistics

MAT 2125 Mid-Term Examination 2015

5-March, 2015. Duration: 80 minutes

Instructor: Barry Jessup

Family Name:______________________

First Name:_______________________

Student number:____________________

1	
2	
3	
4	
5	
(Bonus) 6	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY

1. The correct answer requires reasonable justification written legibly and logically. Proofs and explanations must be clear. Use full and grammatically correct mathematical sentences. Unless otherwise stated, you may use known theorems, but be sure to verify their hypotheses, and wherever possible name the theorems. You must convince me that you know why your solution is correct.
2. Questions 1-5 are worth an equal number of points. Question 6 is a bonus question (so that the maximum on the test is 110%), and **should not be attempted until all parts of questions 1-5 have been completed and checked.** It is *much* more difficult to earn points in the bonus question.
3. Please use the space provided, including the backs of pages if necessary. If you need scrap paper, please ask.
4. You have 80 minutes to complete this exam. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted.
5. Good luck, bonne chance!

1. a) If $\{a_n\}_{n\geq 1}$ is a real sequence, and $a \in \mathbf{R}$, give the definition of

$$\lim_{n\to\infty} a_n = a.$$

Now define a sequence $\{a_n\}_{n\geq 1}$ recursively by

$$a_{n+1} = \begin{cases} 1, & \text{if } n = 1 \\ \dfrac{2a_n}{5} + 3, & \text{if } n \geq 1. \end{cases}$$

b) Prove that $1 \leq a_n < 5$, for all $n \geq 1$

c) Prove that $a_n < a_{n+1}$, for all $n \geq 1$.

d) Prove (using a theorem) that $\{a_n\}_{n\geq 1}$ converges, and find its limit.

2. a) Define what is meant by " a is an accumulation point of $\{a_n\}_{n\geq 1}$."

b) State the Bolzano-Weierstrass theorem for bounded real sequences.

c) If $\{a_n\}_{n\geq 1}$ is a bounded real sequence, define $\limsup_{n\to\infty} a_n$.

d) Let $\{a_n\}_{n\geq 1}$ be a bounded real sequence, and for each $x \in \mathbf{R}$, define $I(x) = \{n \in \mathbf{N} \mid x < a_n\}$, and $J = \{x \mid I(x) \text{ is infinite}\}$. Recall that $\limsup_{n\to\infty} a_n = \sup J$.

Set

$$L = \{l \mid l \text{ is an accumulation point of } \{a_n\}_{n\geq 1}\}.$$

Prove that $\limsup_{n\to\infty} a_n = \sup J = \sup L$, by showing that $s = \sup L$ satisfies the two properties of $\sup J$, as follows:

(i) Using B-W show that if $y \in J$, then there is an accumulation point $l \in L$ with $y \leq l$. (Thus, $y \leq l \leq s$.)

(ii) Let $\varepsilon > 0$. Use the second property of $s = \sup L$ to show that $\exists x \in J$ and $x \in (s-\varepsilon, s]$. (Draw a large picture of the interval $(s-\varepsilon, s]$: you know there is an $l \in L$ in that interval. Now show there must be an $x \in J$ with $x \leq l$, in the same interval ...)

3. a) Let $\{c_n\}_{n\geq 1}$ be a real sequence. Define

$$\text{“The series } \sum_{n\geq 1} c_n \text{ converges.”}$$

Now consider the series

$$\sum_{n\geq 1} (-1)^n \frac{n}{n^2+2}$$

b) Does this series converge?

c) Is this series absolutely convergent?

(In (b) and (c), you may use known theorems, but be sure to verify their hypotheses.)

4. Let A B, and C be subsets of $\mathbf{R}^2$.

a) Define "A is open".

b) State a theorem giving necessary and sufficient conditions for B to be closed *in terms of sequences from B*.

c) Define "C is compact".

d) State a theorem giving necessary and sufficient conditions for C to be compact, different from your answer in (c).

e) If $f : \mathbf{R}^2 \to \mathbf{R}$, give a statement equivalent to "f is continuous on $\mathbf{R}^2$ " in terms of inverse images (under f) of closed sets in $\mathbf{R}$.

f) Prove that $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^4 \leq 1\}$ is compact.

5. Let $A \subseteq \mathbf{R}$, $a \in A$ and $f : A \to \mathbf{R}$.

a) Define

$$\text{“ The function } f \text{ is continuous at a.”}$$

b) Prove that $\forall x \in \mathbf{R}, \quad |x - 1| < 1 \Longrightarrow |x - 3| > 1$.

c) Define $f : [0, 2] \to \mathbf{R}$ by $f(x) = \dfrac{2}{x - 3}$. Prove, from first principles (i.e. use the definition), that f is continuous at 1.

6. (Bonus) Suppose $K \subset \mathbf{R}^n$ is compact, $f : K \to \mathbf{R}$ is continuous on K, and that $\forall u, v \in K$ there exists $p : [0,1] \to K$, continuous on $[0,1]$, such that $p(0) = u$ and $p(1) = v$.

Prove that there exists u_m and $u_M \in K$ such that

$$f(K) = [f(u_m), f(u_M)].$$