MAT 2125 Mid-Term Examination 2015 – Solutions

1. a) If $\{a_n\}_{n\geq 1}$ is a real sequence, and $a \in \mathbf{R}$, give the definition of

$$\lim_{n\to\infty} a_n = a.$$

Solution:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \text{ such that } k \geq N \Rightarrow |a - a_n| < \varepsilon$$

or

$$\forall \varepsilon > 0 \quad \exists N \in \mathbf{N} \text{ such that } \forall k \geq N,\ |a - a_n| < \varepsilon$$

[**2 point(s)**]

Now define a sequence $\{a_n\}_{n\geq 1}$ recursively by

$$a_{n+1} = \begin{cases} 1, & \text{if } n = 1 \\ \dfrac{2a_n}{5} + 3, & \text{if } n \geq 1. \end{cases}$$

b) Prove that $1 \leq a_n < 5$, for all $n \geq 1$

Solution: We proceed by induction on n. The statement is true for $n = 1$, by definition. If $1 \leq a_n < 5$, then $1 < \frac{2}{5} + 3 \leq \frac{2a_n}{5} + 3 < 2 + 3$ so $1 < a_{n+1} < 5$. This closes the induction and establishes the result. [**2 point(s)**]

c) Prove that $a_n < a_{n+1}$, for all $n \geq 1$.

Solution: We could proceed by induction on n, but there's no need. Simply note that as $a_n < 5$, for all $n \geq 1$, we have $-a_n > -5$ for all $n \geq 1$, so

$$a_{n+1} - a_n = \frac{2a_n}{5} + 3 - a_n = -\frac{3}{5}a_n + 3 > -\frac{3}{5}(-5) + 3 = -3 + 3 = 0,$$

for all $n \geq 1$. Hence $a_n < a_{n+1}$, for all $n \geq 1$. [**2 point(s)**]

d) Prove (using a theorem) that $\{a_n\}_{n\geq 1}$ converges, and find its limit.

Solution: Since $\{a_n\}_{n\geq 1}$ is (by (a) and (b)) and increasing sequence which is bounded above, by Bolzano's theorem, $\{a_n\}_{n\geq 1}$ converges to (say) a. [**2 point(s)**]

By taking limits as $n \to \infty$ on both sides of the equation $a_{n+1} = \frac{2a_n}{5} + 3$, we find, by the algebra of limits, that $a = \frac{2a}{5} + 3$, which has the unique solution $a = 5$. Hence $a_n \to 5$. [**2 point(s)**]

2. a) Define what is meant by " a is an accumulation point of $\{a_n\}_{n\geq 1}$." **[2 point(s)]**

b) State the Bolzano-Weierstrass theorem for bounded real sequences.
[2 point(s)]

c) If $\{a_n\}_{n\geq 1}$ is a bounded real sequence, define $\limsup_{n\to\infty} a_n$.
[2 point(s)]

d) Let $\{a_n\}_{n\geq 1}$ be a bounded real sequence, and for each $x \in \mathbf{R}$, define $I(x) = \{n \in \mathbf{N} \mid x < a_n \}$, and $J = \{x \mid I(x) \text{ is infinite}\}$. Recall that $\limsup_{n\to\infty} a_n = \sup J$.

Set

$$L = \{l \mid l \text{ is an accumulation point of } \{a_n\}_{n\geq 1}\}.$$

Prove that $\limsup_{n\to\infty} a_n = \sup J = \sup L$, by showing that $s = \sup L$ satisfies the two properties of $\sup J$, as follows:

(i) Using B-W show that if $y \in J$, then there is an accumulation point $l \in L$ with $y \leq l$. (Thus, $y \leq l \leq s$.)
[2 point(s)]

Solution: If $y \in J$, then $I(y) = \{n \in \mathbf{N} \mid y < a_n \}$ is infinite. Arrange $I(y)$ in increasing order. Define a subsequence of $\{a_n\}_{n\geq 1}$ by $a_{n_k} = a_m$ where m is the k^{th} element of $I(y)$. Since $\{a_n\}_{n\geq 1}$ is bounded (above as well), so is $\{a_{n_k}\}_{k\geq 1}$. By B-W, this will have an accumulation point l which is also an accumulation point of $\{a_n\}_{n\geq 1}$, and since $I(y) = \{n \in \mathbf{N} \mid y < a_n \}$, l will satisfy $y \leq l$. Since $s = \sup L$, we have $y \leq l \leq s$, as required. So s is an upper bound for J.

(ii) Let $\varepsilon > 0$. Use the second property of $s = \sup L$ to show that $\exists x \in J$ and $x \in (s-\varepsilon, s]$. (Draw a large picture of the interval $(s-\varepsilon, s]$: you know there is an $l \in L$ in that interval. Now show there must be an $x \in J$ with $x \leq l$, in the same interval ...)
[2 point(s)]

Solution: Since $s = \sup L$, and $\varepsilon > 0, \exists l \in L$ with $s - \varepsilon < l \leq s$. Since $l \in L$, let $\{a_{n_k}\}_{k\geq 1}$ be a subsequence of $\{a_n\}_{n\geq 1}$ with $\lim_{k\to\infty} = l$. Then, for $\varepsilon' = \dfrac{l-(s-\varepsilon)}{2}$,

$$\exists N \in N s.t. \quad k \geq N \Rightarrow \frac{l+(s-\varepsilon)}{2} = l - \varepsilon' < a_{n_k}.$$

Then, $\{n_k \mid k \geq N\}$ is infinite, since $k \mapsto n_k$ is strictly increasing and hence injective. Hence if $x = \dfrac{l+(s-\varepsilon)}{2}$, then $I(x)$ is infinite, and so $x \in J$. Moreover, $x \in (s-\varepsilon, s]$. So, $s = \sup J$.

3. a) Let $\{c_n\}_{n\geq 1}$ be a real sequence. Define

$$\text{“The series } \sum_{n\geq 1} c_n \text{ converges.”}$$

[2 point(s)]

Now consider the series

$$\sum_{n\geq 1}(-1)^n \frac{n}{n^2+2}$$

b) Does this series converge? **[5 point(s)]**

Solution: This is an alternating series. Define $a_n = \frac{n}{n^2+2}$. Note that $\forall n \geq 1, a_n > 0$, and moreover, for $n \geq 2$, $0 < a_n = \frac{n}{n^2+2} = \frac{1}{n+\frac{2}{n}} < \frac{1}{2n} \to 0$, so by the Squeeze theorem, $a_n \to 0$.

Further, it is clear that for $n \geq 2$, $(n+1) + \frac{2}{n+1} > n + \frac{2}{n}$ and so $a_{n+1} = \dfrac{1}{(n+1)+\frac{2}{n+1}} < \dfrac{1}{n+\frac{2}{n}} = a_n$, for for $n \geq 3$.

Hence $\{a_n\}_{n\geq 1}$ is a positive, decreasing sequence with limit zero. By Leibniz's theorem, $\sum_{n\geq 1}(-1)^n a_n$ converges.

c) Is this series absolutely convergent? **[3 point(s)]**

Solution: No, by comparison, since for $n \geq 2$, $a_n = \dfrac{1}{n+\frac{2}{n}} \geq \frac{1}{n+1} \geq 0$, and we know $\sum_{n\geq 1} \dfrac{1}{n+1}$ diverges.

(In (b) and (c), you may use known theorems, but be sure to verify their hypotheses.)

4. Let A B, and C be subsets of $\mathbf{R}^2$.

a) Define "A is open".
[1 point(s)]

b) State a theorem giving necessary and sufficient conditions for B to be closed *in terms of sequences from* B. **[1 point(s)]**

c) Define "C is compact". **[2 point(s)]**

d) State a theorem giving necessary and sufficient conditions for C to be compact, different from your answer in (c).
[1 point(s)]

e) If $f : \mathbf{R}^2 \to \mathbf{R}$, give a statement equivalent to "f is continuous on $\mathbf{R}^2$" in terms of inverse images (under f) of closed sets in $\mathbf{R}$. **[2 point(s)]**

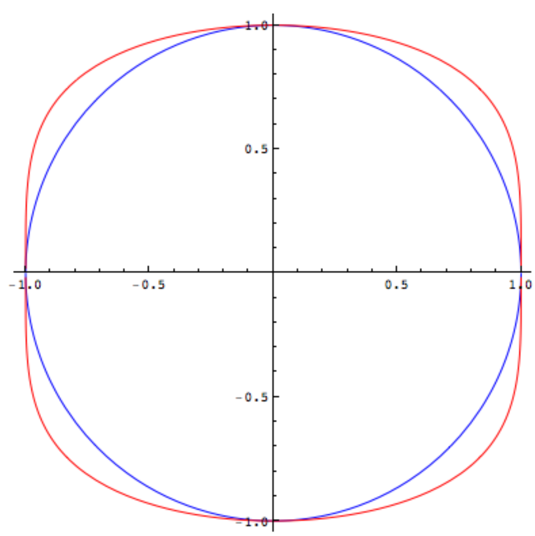
f) Prove that $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^4 \leq 1\}$ is compact.
[3 point(s)]

Solution: Define $f : \mathbf{R}^2 \to \mathbf{R}$ by $f(x, y) = x^2 + y^4$. We know that f is continuous, being a sum of products of compositions of cts functions. We also know that $[0, 1]$ is a closed set in $\mathbf{R}$, so $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^4 \leq 1\} = \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x^2 + y^4 \leq 1\} = f^{-1}([0, 1])$ is closed by (e).

Moreover, $x^2 + y^4 \leq 1 \Rightarrow x^2 \leq 1$ and $y^4 \leq 1 \Rightarrow x^2 \leq 1$ and $y^2 \leq 1 \Rightarrow x^2 + y^2 \leq 2$, so D is bounded. (The best upper bound is 1.25, but we don't need this sort of precision.)

Hence D is compact.

(In the diagram below, the blue plot has equation $x^2 + y^2 = 1$, while the red plot has equation $x^2 + y^4 = 1$. So $x^2 + y^2 \leq 1 \not\Rightarrow x^2 + y^4 \leq 1$.)



5. Let $A \subseteq \mathbf{R}$, $a \in A$ and $f : A \to \mathbf{R}$.

a) Define

“ The function f is continuous at a.”

[3 point(s)]

b) Prove that $\forall x \in \mathbf{R}, \quad |x-1| < 1 \Longrightarrow |x-3| > 1.$ **[2 point(s)]**

Solution: $|x-1| < 1 \iff -1 < x-1 < 1 \iff -3 < x-3 < -1 \Rightarrow |x-3| > 1.$

c) Define $f : [0,2] \to \mathbf{R}$ by $f(x) = \dfrac{2}{x-3}$. Prove, from first principles (i.e. use the definition), that f is continuous at 1.
[5 point(s)]

Solution: Let $\varepsilon > 0$ and choose $\delta = \min\{1, \varepsilon\}$. Note from part (b) that that

$$|x-1| < \delta \Rightarrow \frac{1}{|x-3|} < 1.$$

Then

$$|x-1| < \delta \Rightarrow |f(x) - f(1)| = \left|\frac{2}{x-3} - (-1)\right| = \left|\frac{x-1}{x-3}\right| < |x-1| < \varepsilon$$

Hence f is continuous at 1.

6. (Bonus) Suppose $K \subset \mathbf{R}^n$ is compact, $f : K \to \mathbf{R}$ is continuous on K, and that $\forall u, v \in K$ there exists $p : [0,1] \to K$, continuous on $[0,1]$, such that $p(0) = u$ and $p(1) = v$.

Prove that there exists u_m and $u_M \in K$ such that

$$f(K) = [f(u_m), f(u_M)].$$

[5 point(s)]

Solution: We know from the DGD on 4-Mar that there exists u_m and $u_M \in K$ such that $\forall x \in K, \quad f(u_m) \leq f(x) \leq f(u_M)$, i.e. , $f(K) \subseteq [f(u_m), f(u_M)]$. To finish we need only show equality in the set inclusion.

So let $y \in [f(u_m), f(u_M)]$. Since K is path-connected, there exists $p : [0,1] \to K$, continuous on $[0,1]$, such that $p(0) = u_m$ and $p(1) = u_M$. Then, the composition $f \circ p : [0,1] \to [f(u_m), f(u_M)]$, begin the composiiton of continuous functions, is also continuous on $[0,1]$.

By the IVT applied to $f \circ p$ on $[0,1]$, for the aforementioned $y \in [f(u_m), f(u_M)]$, there exists $t \in [0,1]$ with $f \circ p(t) = y$. If we set $k = p(t)$, then $k \in K$, and $f(k) = y$.

Hence, $f(K) = [f(u_m), f(u_M)]$.