uOttawa University of Ottawa

Faculty of Science »
Department of mathematics and statistics

MAT 2125 Final Examination 2015

28-April, 2015. Duration: 3 hours

Instructor: Barry Jessup

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(Bonus) 10	
Total	

Family Name:______________________

First Name:________________________

Student number:____________________

PLEASE READ THESE INSTRUCTIONS CAREFULLY

1. The correct answer requires reasonable justification written legibly and logically. Proofs and explanations must be clear. Use full and grammatically correct mathematical sentences. Unless otherwise stated, you may use known theorems, but be sure to verify their hypotheses, and wherever possible name the theorems. You must convince me that you know why your solution is correct.
2. Questions 1-9 are worth an equal number of points. Question 10 is a bonus question (so that the maximum on the test is 110%), and **should not be attempted until all parts of questions 1-9 have been completed and checked.** It is much more difficult to earn points in the bonus question.
3. Please use the space provided, including the backs of pages if necessary. If you need scrap paper, please ask.
4. You have 3 hours minutes to complete this exam. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted. All implanted cyber devices not necessary for life-support must be disabled at the beginning of the exam.
5. Good luck, bonne chance!

1. Let $A \subseteq \mathbf{R}$. We say that a real number $M \in \mathbf{R}$ is a *maximum of* A, if M is an upper bound for A and $M \in A$.

a) State necessary and sufficient conditions for the supremum of A, $\sup A$ to exist.

b) Given an example of a set A where $\sup A$ exists but $\max A$ does not. (You do not need to prove that your example is a good one. Just be sure it is.)

c) Prove that if $\max A$ exists, then so does $\sup A$ and $\sup A = \max A$.

d) Now suppose $A \subset \mathbf{Z}$ is a non-empty subset of the integers which is bounded above. Prove that $\sup A$ is an integer.

2. Let $\{a_n\}_{n\geq 1}$ be a real sequence.

a) Define what is meant by "$\{a_n\}_{n\geq 1}$ is a Cauchy sequence."

b) Give a statement, different from the definition, which is equivalent to "$\{a_n\}_{n\geq 1}$ is a Cauchy sequence."

c) Define what is meant by "b is an accumulation point of $\{a_n\}_{n\geq 1}$."

d) State the Bolzano-Weierstrass theorem for real sequences.

e) Suppose $\{a_n\}_{n\geq 1} \subset [a, b]$ for some real numbers $a < b$. Prove that the sequence

$$\left\{\frac{a_n}{n}\right\}_{n\geq 1}$$

is Cauchy.

3. a) Let $\{b_n\}_{n\geq 1}$ be a real sequence. Define

$$\text{“The series } \sum_{n=1}^{\infty} b_n \text{ is absolutely convergent.”}$$

b) Prove by induction for that $2^n \geq 2^{n-1} + n$, $\forall n \in \mathbf{N}, n \geq 1$.

c) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n - n}$ is absolutely convergent.

d) Give an example of a convergent series which is not absolutely convergent.

e) Give an example of an absolutely convergent series which is not convergent, or give reasons it is impossible to do so.

(In (c)– (e), you may use known tests and theorems, but be sure to verify their hypotheses.)

4. Let $A \subseteq \mathbf{R}$, $a \in A$ and $f : A \to \mathbf{R}$.

a) Define

$$\text{“ The function } f \text{ is continuous at a.”}$$

b) Prove that $\forall x \in \mathbf{R}, \quad |x-1| < 1 \Longrightarrow 1 + x^2 > 1.$

c) Prove that $\forall x \in \mathbf{R}, \quad |x-1| < 1 \Longrightarrow |1+x| < 3.$

d) Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = \dfrac{1}{1+x^2}.$

Prove, using the definition (i.e., using “$\varepsilon - \delta$, etc.”) that f is continuous at 1.

e) Give an example of a function $g : (0,1] \to \mathbf{R}$ which is continuous on $(0,1]$ but is not *uniformly continous* on $(0,1]$.

5. Let A and C be subsets of $\mathbf{R}^p$.

a) Define "A is an open subset of $\mathbf{R}^p$."

b) Prove from the definition that if A and C are open, then so is their intersection $A \cap C$.

c) Give two different characterizations of "C is compact", one of which is in terms of sequences in C.

d) Suppose $A \neq \emptyset$, $a \in A$ and that $X = \{r \mid r \in \mathbf{R},\ r > 0,\ \text{and } \mathbf{B}(a, r) \subset A\}$ is unbounded. Prove that $A = \mathbf{R}^p$.

e) Now suppose $p = 1$, and that $A \subseteq \mathbf{R}$ is both open and closed. Prove carefully that $A = \emptyset$ or $A = \mathbf{R}$.

6. Let $f, g, h : [-1, 1] \to \mathbf{R}$ be three functions.

a) If $c \in (-1, 1)$, define "f is differentiable at c."

b) If g is continuous at 0, and $f(x) = xg(x), \forall x \in [-1, 1]$, prove that f is differentiable at 0. (We do *not* assume that g is differentiable anywhere.)

c) State the the Mean Value Theorem *for derivatives*.

d) Now suppose h is continuous at 0, and is differentiable on $(-1, 0)$ and $(0, 1)$. If $\lim_{x \to 0} h'(x)$ exists, prove that h *is also* differentiable at 0 and that $\lim_{x \to 0} h'(x) = h'(0)$. (Hint: Use the MVT.)

7. Let A be a subset of $\mathbf{R}$, and $f_n : \mathbf{R} \to \mathbf{R}$ a sequence of functions . Now consider the series $\sum_{n=0}^{\infty} f_n(x)$.

a) If $f : A \to \mathbf{R}$ is a function, define " $\sum_{n=0}^{\infty} f_n$ converges uniformly on A to f."

b) State the Weierstrass M-test.

c) Prove that $\forall K \in \mathbf{R}$, the series $\sum_{n=0}^{\infty} \frac{K^{2n+1}}{(n+1)!}$ converges.

Now define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(n+1)!}.$$

d) Explain briefly (use a theorem!) why f is differentiable on $\mathbf{R}$, and give its derivative.

e) If $a \in \mathbf{R}$, find $\int_0^a f$, carefully justifying your steps (by use of theorems, if necessary).

8. Define $f : [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 2 & \text{for } x \in [0, \frac{1}{2}) \\ 0 & \text{for } x = \frac{1}{2} \\ -1 & \text{for } x \in (\frac{1}{2}, 1] \end{cases}$$

a) State the Intermediate Value Theorem.

b) Prove that f is not continuous on $[0,1]$. You may use a theorem.

c) Carefully state a necessary and sufficient condition for f to be Riemann (-Darboux) integrable in terms of upper sums $U(f,P)$ and lowers sums $L(f,P)$, where P denotes a partition of $[0,1]$. (You may give the definition or an equivalent condition.)

d) For $n \in \mathbf{N}, n \geq 3$, let P_n be the partition $P_n = \{0, \frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}, 1\}$. Find $U(f, P_n)$ and $L(f, P_n)$, for all $n \geq 3$.

e) Use your result in (d) and your response in (c) to prove that f is integrable, and find $\displaystyle\int_0^1 f$.

9. Define a function $f : \mathbf{R} \to \mathbf{R}$ by by $f(x) = \displaystyle\int_0^x \frac{1}{1+4t^2}\, dt$.

a) *Briefly* explain why f is differentiable on $\mathbf{R}$. (Use theorems.)

b) State the Mean Value Theorem *for integrals*.

c) Prove that if we define $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = 2f(\frac{x}{2})$, then g is strictly increasing on $\mathbf{R}$.

d) Find $g'(x)$.

Now denote $g(1) = p$. Let

$$h : [0, p] \to [0, 1]$$

be the inverse function for g, which we know exists by parts (c) and (d), and define $k : [0, p] \to \mathbf{R}$ by

$$k(x) = \sqrt{1 + h^2(x)}, \quad \forall x \in [0, p].$$

e) *Briefly* explain why h and k are differentiable on $(0, p)$, and show that

$$h'(x) = k^2(x),$$

and

$$k'(x) = h(x)k(x),$$

for all $x \in (0, p)$. (Use theorems!)

10. (Bonus) Suppose $\{F_n \mid n \in \mathbf{N}, n \geq 1\}$ is a sequence of closed subsets of $\mathbf{R}^p$, such that

(i) F_1 is bounded, and

(ii) $\displaystyle\bigcap_{i=1}^{n} F_i \neq \emptyset$, for all $n \in \mathbf{N}$.

Prove that $\displaystyle\bigcap_{i=1}^{\infty} F_i \neq \emptyset$.