uOttawa University of Ottawa

Faculty of Science »

Department of mathematics and statistics

MAT 2125 Mid-Term Examination 2014

February 27, 2014. Duration: 80 minutes

Instructor: Barry Jessup

Family Name:______________________

First Name:________________________

Student number:____________________

1	
2	
3	
4	
5	
(Bonus) 6	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY

1. The correct answer requires reasonable justification written legibly and logically. Proofs and explanations must be clear. Use full and grammatically correct mathematical sentences. Unless otherwise stated, you may use known theorems, but be sure to verify their hypotheses, and wherever possible name the theorems. You must convince me that you know why your solution is correct.
2. Questions 1-5 are worth an equal number of points. Question 6 is a bonus question (so that the maximum on the test is 110%), and **should not be attempted until all parts of questions 1-5 have been completed and checked.** It is much more difficult to earn points in the bonus question.
3. Please use the space provided, including the backs of pages if necessary. If you need scrap paper, please ask.
4. You have 80 minutes to complete this exam. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted.
5. Good luck, bonne chance!

1. Let A be a subset of $\mathbf{R}$.

a) Define "A is bounded below".

b) Define "$\inf A$".

c) State necessary and sufficient conditions for the existence of $\inf A$.

d) Suppose $\inf A$ exists, and let $t \in \mathbf{R}$ be a fixed real number. Define a subset B of $\mathbf{R}$ by

$$B = \{t + a \mid a \in A\}.$$

Prove that $\inf B$ exists and is equal to $t + \inf A$.

2. a) Define what is meant by "$\{a_n\}_{n\geq 1}$ is a Cauchy sequence".

b) If $\{a_n\}_{n\geq 1}$ is a bounded real sequence, define $\limsup_{n\to\infty} a_n$.

Parts (b) and (c) concern a sequence $\{b_n\}_{n\geq 1}$ defined by

$$b_n = \begin{cases} 2014 - \frac{1}{n}, & \text{if } n \text{ is even} \\ -10, & \text{if } n \text{ is odd.} \end{cases}$$

c) Prove that $\{b_n\}_{n\geq 1}$ is bounded but is *not* Cauchy.

d) Find a convergent subsequence of $\{b_n\}_{n\geq 1}$ that is neither $\{b_{2n}\}_{n\geq 1}$, nor $\{b_{2n+1}\}_{n\geq 1}$.

3. a) Let $\{c_n\}_{n\geq 1}$ be a real sequence. Define

$$\text{“The series } \sum_{n\geq 1} c_n \text{ converges.”}$$

Now consider the series

$$\sum_{n\geq 1} (-1)^n \frac{n+1}{n^2}$$

b) Does this series converge?

c) Is this series absolutely convergent?

(In (b) and (c), you may use known theorems, but be sure to verify their hypotheses.)

4. Let A and B be two subsets of $\mathbf{R}^2$.

a) Define "A is open ".

b) Now suppose A and B are both open sets. Prove that

$$A \cap B = \{v \in \mathbf{R}^2 \mid v \in A \text{ and } v \in B\}$$

is also open.

c) State a theorem giving necessary and sufficient conditions for B to be closed *in terms of sequences from* B.

d) Give and example of a subset of $\mathbf{R}^2$ which is closed but is not compact. (You do not need to justify your answer. It simply needs to be correct.)

5. Let $A \subseteq \mathbf{R}$, $a \in A$ and $f : A \to \mathbf{R}$.

a) Define

“ The function f is continuous at a.”

b) Prove that $\forall x \in \mathbf{R}, \quad |x-2| < 1 \Longrightarrow |x+4| > 5$.

c) Define $f : \mathbf{R} \setminus \{-4\} \to \mathbf{R}$ by $f(x) = \dfrac{1}{x+4}$.

Prove from first principles (i.e. using an “$\varepsilon - \delta$” argument) carefully that f is continuous at $a = 2$.

6.

(Bonus) Suppose that $\{a_n\}_{n\geq 1}$ is a sequence of real numbers.

(i) If $\{a_n\}_{n\geq 1}$ an *increasing* sequence, prove that if $\{a_n\}_{n\geq 1}$ diverges then every subsequence $\{a_{n_k}\}_{k\geq 1}$ of $\{a_n\}_{n\geq 1}$ also diverges.

(ii) Give an example of a sequence $\{a_n\}_{n\geq 1}$ that does not converge, but which has a convergent subsequence.