2125 W14 Mid Term Sol'ns 5/3

**1.** Let $A$ be a subset of $\mathbf{R}$.

a) Define "$A$ is bounded below".

b) Define "inf $A$".

c) State necessary and sufficient conditions for the existence of inf $A$.

d) Suppose inf $A$ exists, and let $t \in \mathbf{R}$ be a fixed real number. Define a subset $B$ of $\mathbf{R}$ by

$$B = \{t + a \mid a \in A\}.$$

Prove that inf $B$ exists and is equal to $t + \inf A$.

(1) a) $A$ is bounded below if $\exists M \geq 0$ st $\forall a \in A$, $a \leq M$.

b) If $l \in \mathbb{R}$, $l = \inf A$ if (I) $\forall a \in A$, $l \leq a$

(1.5) (II) $\forall \varepsilon > 0$, $\exists a \in A$ s.t. $l \leq a < l + \varepsilon$

(1) c) For $\inf A$ to exist, $A$ must be non-empty and bounded below.

d) Let $l = \inf A$. We show $l+t = \inf B$:

(1) (I)' $\forall b \in B$, $\exists a \in A$ st $b = t + a$. But by (I)' above, $\forall a \in A$, $l \leq a$.

Hence, $\forall b \in B$, $l + t \leq (t + a =) b$.

(2) (II)' Let $\varepsilon > 0$ and choose $a \in A$ s.t. $l \leq a < l + \varepsilon$. Then

$l + t \leq a + t < (l + t) + \varepsilon$. But $a + t \in B$, so $\exists b \in B$ st

$l + t \leq b < (l + t) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary,

$\inf A + t = l + t = \inf B$.



+0.5 well written

4

[2] **2.** a) Define what is meant by "$\{a_n\}_{n\geq 1}$ is a Cauchy sequence".

[1½] b) If $\{a_n\}_{n\geq 1}$ is a bounded real sequence, define $\limsup\limits_{n\to\infty} a_n$.

Parts (~~b~~ c) and (~~c~~ d) concern a sequence $\{b_n\}_{n\geq 1}$ defined by

$$b_n = \begin{cases} 2014 - \frac{1}{n}, & \text{if } n \text{ is even} \\ -10, & \text{if } n \text{ is odd.} \end{cases}$$

[2½] c) ~~Find $\limsup\limits_{n\to\infty} b_n$. (You must justify your answer.)~~ Show that $\{b_n\}_{n\geq 1}$ is bdd but is not Cauchy.

[1] d) Find a convergent subsequence of $\{b_n\}_{n\geq 1}$ that is neither $\{b_{2n}\}_{n\geq 1}$, nor $\{b_{2n+1}\}_{n\geq 1}$.

0 a) The sequence $\{a_n\}_{n\geq 1}$ is Cauchy if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n, p \geq N$,
$|a_n - a_p| < \varepsilon$.

0

1¼ b) First, for $n \geq 1$, $n \in \mathbb{N}$, define $t_n = \sup\{a_k \mid k \geq n\}$ ¾ [Then we know $\{t_n\}_{n\geq 1}$ is a decreasing sequence which is bounded below,

so $\exists \limsup\limits_{n\to\infty} a_n := \lim\limits_{n\to\infty} t_n$ ¾

0

[2½] c) Since $-10 \leq b_n < 2014$, $\forall n \geq 1$, $\{b_n\}$ is bdd. (Indeed, $|b_n| < 2014, \forall n \geq 1$). 1

However, for $\varepsilon = 1$, $\forall N \in \mathbb{N}$, $|b_{2N} - b_{2N+1}| = 2014 - \frac{1}{n} + 10 > 2023 \geq 1$.

1.5 Hence $\{b_n\}$ is not cauchy. ($\frac{1}{2}$ liminf, $\frac{1}{2}$ lim sup, $\frac{1}{2}$ lim inf ≠ lim sup ⇒ not Cauchy)

0

1 d) Define $b_{n_k} = b_{4k}$ (i.e. $n_k = 4k$). Then

$\{b_{4k}\}_{k\geq 1} = \{2014 - \frac{1}{4k}\}_{k\geq 1}$, which converges to 2014.

or Define $b_{n_k} = b_{4k+1}$, $k \geq 1$. Then

$\{b_{4k+1}\}_{k\geq 1} = \{-10\}_{k\geq 1}$, so $b_{4k+1} \to -10$

**3.** a) Let $\{c_n\}_{n\geq 1}$ be a real sequence. Define

[2]

$$\text{“The series } \sum_{n\geq 1} c_n \text{ converges.”}$$

Now consider the series

$$\sum_{n\geq 1} (-1)^n \frac{n+1}{n^2}$$

[3] b) Does this series converge?

[2] c) Is this series absolutely convergent?

(In (b) and (c), you may use known theorems, but be sure to verify their hypotheses.)

a) The series $\sum_{n\geq 1} c_n$ converges iff the sequence $\left\{ \sum_{k=1}^{n} c_k \right\}_{n\geq 1}$ (of

2

partial sums) converges

---

b) Yes, this series converges by Leibniz's thm. Let $a_n = \frac{n+1}{n^2}$

Then 1 (I) $0 \leq a_n, \forall n \geq 1$

1 (II) $a_{n+1} = \frac{n+2}{(n+1)^2} = \frac{n+1}{(n+1)^2} + \frac{1}{(n+1)^2} = \frac{1}{n+1} + \frac{1}{(n+1)^2} < \frac{1}{n} + \frac{1}{n^2} = \frac{n+1}{n^2} = a_n$,

$\forall n \geq 1$, so $a_{n+1} < a_n \; \forall n \geq 1$

1 (III) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{1}{n^2} = 0 + 0 = 0.$

(or: $0 \leq a_n \leq \frac{n+1}{(n+1)^2} = \frac{1}{n+1}$, and $\lim_{n\to\infty} \frac{1}{n+1} = 0$; Use squeeze thm.)

Hence, by Leibniz's thm, $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

---

c) No: As $a_n = \frac{n+1}{n^2} \overset{1}{>} \frac{n}{n^2} = \frac{1}{n} \geq 0$, and we know $\sum_{n=1}^{\infty} \frac{1}{n}$
(see (b))



diverges, by comparison, $\sum_{n\geq 1} a_n = \sum_{n\geq 1} \left| (-1)^n \frac{n+1}{n^2} \right|$ diverges. That is,

$\sum_{n\geq 1} (-1)^n \frac{n+1}{n^2}$ does not converge absolutely. 1

**4.** Let $A$ and $B$ be two subsets of $\mathbf{R}^2$.

a) Define "$A$ is open".

b) Now suppose $A$ and $B$ are both open sets. Prove that

$$A \cap B = \{v \in \mathbf{R}^2 \mid v \in A \text{ and } v \in B\}$$

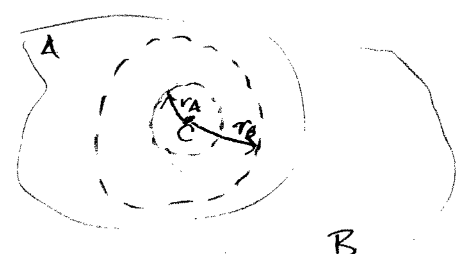
is also open.

c) State a theorem giving necessary and sufficient conditions for $B$ to be closed *in terms of sequences from $B$.*

d) Give and example of a subset of $\mathbf{R}^2$ which is closed but is not compact. (You do not need to justify your answer. It simply needs to be correct.)

a) A subset $A \subseteq \mathbb{R}^2$ is open if $\forall a \in A \; \exists r > 0$ s.t.
1 $B(a,r) \subseteq A$


b) Let $c \in A \cap B$. Since $A$ is open, $\exists r_A > 0$ 1 st. $B(c, r_A) \subseteq A$. Similarly, since $B$ is open and $c \in B$, $\exists r_B > 0$ 1 s.t. $B(c, r_B) \subseteq B$.




Let $r = \min(r_A, r_B)$. Then $B(c,r) \subseteq B(c, r_A) \subseteq A$ and
1 $B(c,r) \subseteq B(c, r_B) \subseteq B$.

Hence $B(c,r) \subseteq A \cap B$. 1½ Since $c \in A \cap B$ was arbitrary, $A \cap B$ is open.

c) A subset $B$ is closed if every convergent sequence in $B$ has its
1½ limit in $B$

d) $\mathbb{R}^2 \subseteq \mathbb{R}^2$ is closed (see c!) but is not compact as
1 $\mathbb{R}^2$ is not bdd (eg $(n,0) \in \mathbb{R}^2$, $\forall n \in \mathbb{N}$, and $\|(n,0)\| = n, \forall n$)

**5.** Let $A \subseteq \mathbf{R}$, $a \in A$ and $f : A \to \mathbf{R}$.

a) Define

"The function $f$ is continuous at a."

b) Prove that $\forall x \in \mathbf{R}, \quad |x-2| < 1 \Longrightarrow |x+4| > 5$.

c) Define $f : \mathbf{R} \setminus \{-4\} \to \mathbf{R}$ by $f(x) = \dfrac{1}{x+4}$.

Prove from first principles (i.e. using an "$\varepsilon - \delta$" argument) carefully that $f$ is continuous at $a = 2$.

d) Sketch the graph of $f$, i.e., the set of points $\{(x, f(x)) \mid x \in \mathbf{R} \setminus \{-4\}\}$. Label your axes and include any asymptotes. (You do not need to analyse local extrema or concavity, or that prove your claimed asymptotes are correct.)

a) The function $f$ is cts at $a \in A$ if
(2) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x, \ |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

b) Note that $|x+4| = |6 - (2-x)| \geq |6| - |x-2| > 6 - 1 = 5$ is
(1) true $\forall x \in \mathbb{R}$; i.e. $\forall x, \ |x+4| > 5 \quad (*)$

(1)

c) Note that $\left| f(x) - \frac{1}{6} \right| = \left| \frac{1}{x+4} - \frac{1}{6} \right| = \left| \frac{6-x-4}{6(x+4)} \right| = \frac{|x-2|}{6|x+4|}$.

(1)

Now let $\varepsilon > 0$. Choose $\delta = \min(1, 30\varepsilon)$. Then, $\forall x \ (x \neq -4)$ with $|x-2| < \delta$, we have $|x-2| < 1$ (so $(*)$ holds) and

(1/2)

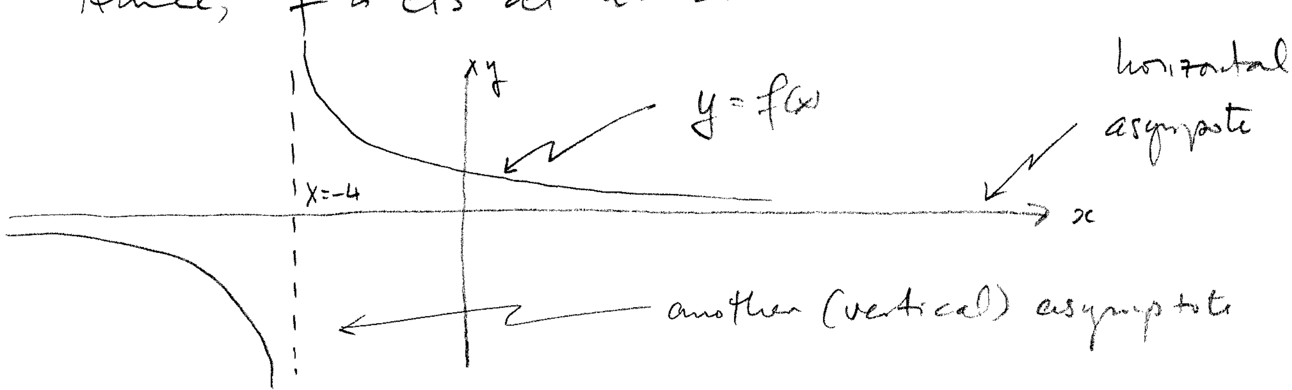


$$\left| f(x) - \frac{1}{6} \right| = \frac{|x-2|}{6|x+4|} < \frac{|x-2|}{6.5} = \frac{|x-2|}{30} < \frac{30\varepsilon}{30} = \varepsilon.$$

(1) (1/2) - well written.

Hence, $f$ is cts at $a = 2$.

d)




7

**6.**

(Bonus) Suppose that $\{a_n\}_{n\geq 1}$ is a sequence of real numbers.

(i) If $\{a_n\}_{n\geq 1}$ an *increasing* sequence, prove that if $\{a_n\}_{n\geq 1}$ diverges then every subsequence $\{a_{n_k}\}_{k\geq 1}$ of $\{a_n\}_{n\geq 1}$ also diverges.

(ii) Give an example of a sequence $\{a_n\}_{n\geq 1}$ that does not converge, but which has a convergent subsequence.

(i) Since we know every increasing sequence converges $\Leftrightarrow$ it is bounded above, in this case we know that $\{a_n\}_{n\geq 1}$ 1/2 is not bdd above. Hence, $\forall m \in \mathbb{N}$, $\exists N$ st. $a_N > m$ $(*)$

Now let $\{a_{n_k}\}_{k\geq 1}$ be a subsequence. It is also increasing and so it suffices to show it is not 1/2 bdd above. Let

1/2 $m \in \mathbb{N}$. By $(*)$, $\exists N \in \mathbb{N}$ st $a_N > m$ But as $n_k \geq k$, $\forall k \geq 1$, $n_N > N$ and so (as $a_n \uparrow$) $a_{n_N} > a_N > m$

1

Hence $\{a_{n_k}\}_{n\geq 1}$ also diverges 1/2

(ii) Let $a_n = (-1)^n$, $n \geq 1$. Then $\{a_n\}_{n\geq 1}$ does not converge (Let $\varepsilon = 1$ and $N \in \mathbb{N}$; choose $n, n+1 \geq N$. Then $|a_{n+1} - a_n| = 2 \geq 1 = \varepsilon$. Hence $\{a_n\}_{n\geq 1}$ is not Cauchy and so cannot converge) However, $\{a_{2n}\}_{n\geq 1} = \{1\}_{n\geq 1}$ and so $a_{2n} \to 1$.

$\frac{1}{2} + \frac{1}{2}$