uOttawa University of Ottawa

Faculty of Science »
Department of mathematics and statistics

MAT 2125 Final Examination 2014

April, 2014. Duration: 3 hours

Instructor: Barry Jessup

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(Bonus) 10	
Total	

Family Name:______________________

First Name:________________________

Student number:____________________

PLEASE READ THESE INSTRUCTIONS CAREFULLY

1. The correct answer requires reasonable justification written legibly and logically. Proofs and explanations must be clear. Use full and grammatically correct mathematical sentences. Unless otherwise stated, you may use known theorems, but be sure to verify their hypotheses, and wherever possible name the theorems. You must convince me that you know why your solution is correct.
2. Questions 1-9 are worth an equal number of points. Question 10 is a bonus question (so that the maximum on the test is 110%), and **should not be attempted until all parts of questions 1-9 have been completed and checked.** It is much more difficult to earn points in the bonus question.
3. Please use the space provided, including the backs of pages if necessary. If you need scrap paper, please ask.
4. You have 3 hours minutes to complete this exam. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted. All implanted cyber devices not necessary for life-support must be disabled at the beginning of the exam.
5. Good luck, bonne chance!

1. Let $A \subseteq \mathbf{R}$.

a) If $s \in \mathbf{R}$, define what is meant by "s is the supremum of A", i.e., $s = \sup A$.

b) If $l \in \mathbf{R}$, define what is meant by "l is the infimum of A", i.e., $l = \inf A$.

c) State necessary and sufficient conditions for both $\sup A$ *and* $\inf A$ to exist.

Now suppose both $l = \inf A$, and $s = \sup A$ exist.

d) Prove that

$$(\forall \varepsilon > 0,\ |s - l| < \varepsilon) \Longrightarrow s = l.$$

e) Prove that $\inf A \leq \sup A$, and that equality holds if and only if $A = \{x\}$ for some $x \in \mathbf{R}$.

2. a) Let $\{b_n\}_{n\geq 1}$ be a real sequence. Give the Cauchy criterion which is equivalent to:

$$\text{“The series } \sum_{n=1}^{\infty} b_n \text{ converges.”}$$

b) Prove that if $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} b_n = 0$.

c) Give an example to show that the converse of the statement in (b) is not true. (No justification is necessary for the example.)

d) Prove that the series $\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{n^2+1}$ converges, but is not absolutely convergent.

(In (d), you may use known tests and theorems, but be sure to verify their hypotheses.)

3. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be real sequences, and $a \in \mathbf{R}$.

a) Define what is meant by "$\lim_{n\to\infty} a_n = a$."

b) Suppose $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ satisfy

(I) $\forall n \geq 1,\ a_n > 0,\ b_n > 0$, and

(II) $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ exists and is non-zero.

Prove that

$$\sum_{n=1}^{\infty} a_n \text{ converges} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} b_n \quad \text{converges.}$$

4. Let $A \subseteq \mathbf{R}^n$, and suppose $f, g : A \to \mathbf{R}$.

a) Give the definition of "f is uniformly continuous on A."

b) Prove that if f and g are uniformly continuous, and are both bounded on A, then fg is uniformly continuous on A.

c) Let $f : (0,1) \to \mathbf{R}$ be defined by $f(x) = \dfrac{1}{x}, \quad \forall x \in (0,1)$. Explain briefly why f is continuous on $(0,1)$, and that $\left\{\dfrac{1}{n}\right\}_{n \geq 1} \subset (0,1)$ is indeed Cauchy.

Now prove that $\{f\left(\dfrac{1}{n}\right)\}_{n \geq 1} \subset (0,1)$ is *not* Cauchy.

5. Let A and B be subsets of $\mathbf{R}^2$, and let $f : \mathbf{R}^2 \to \mathbf{R}$.

a) Give a characterization of "A is closed" in terms of sequences $\{a_n\}_{n\geq 1} \subseteq A$.

b) Give a 3 characterizations of "B is compact":
 (i) in terms of sequences $\{a_n\}_{n\geq 1} \subseteq B$,
 (ii) in terms of open covers of B, and
 (iii) another, different from those already given in (i) and (ii).

c) Give a characterization of "f is continuous on $\mathbf{R}^2$" in terms of inverse images (by f) of open sets in $\mathbf{R}$.

Now let $C = \{(x, y) \in \mathbf{R}^2 \mid xy > 1\}$.

d) Prove that C is open. (*Hint: Consider $f(x, y) = xy$, briefly explain why f is continuous, and use (c).*)

6. Suppose $A \subseteq \mathbf{R}$, $\{f_n\}_{n\geq 1}$ is a sequence of continuous functions $f_n : A \to \mathbf{R}$, and $f : A \to \mathbf{R}$.

a) Give the definition of "$\{f_n\}_{n\geq 1}$ converges pointwise to f on A."

b) Give the definition of "$\{f_n\}_{n\geq 1}$ converges uniformly to f on A."

c) Suppose $\{f_n\}_{n\geq 1}$ converges uniformly on A to f. Prove carefully that f is continuous on A.

d) Suppose (for this part only) that $A = [-1, 1]$, and that $\{f_n\}_{n\geq 1}$ converges uniformly to f on A. Is f *uniformly* continuous on A? You must give reasons for your answer. (Stating a theorem would be sufficient.)

7. Define $f : [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, \frac{1}{3}) \\ 0 & \text{for } x = \frac{1}{3} \\ 1 & \text{for } x \in (\frac{1}{3}, 1] \end{cases}$$

a) Carefully state a necessary and sufficient condition for f to be Riemann (—Darboux) integrable in terms of upper sums $U(f,P)$ and lowers sums $L(f,P)$, where P denotes a partition of $[0,1]$. (You may give the definition, or an equivalent condition.)

b) For $n \in \mathbf{N}, n \geq 4$, let P_n be the partition $P_n = \{0, \frac{1}{3} - \frac{1}{n}, \frac{1}{3} + \frac{1}{n}, 1\}$. Find $U(f, P_n)$ and $L(f, P_n)$, for all $n \geq 4$.

c) Use your result in (b) and your response in (a) to prove directly that f is integrable on $[0,1]$, and find $\int_0^1 f$.

8. Consider the series $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

a) Prove that $\forall K > 0$, the series above converges uniformly on $[-K, K]$.

Now define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

b) Explain briefly why f is continuous on $\mathbf{R}$.

Now define $g : \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \int_0^x f, \qquad \forall x \in \mathbf{R}$$

c) Prove carefully that

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

d) Prove that $\forall r \in \mathbf{R}$ there exists a unique $s \in \mathbf{R}$ with $g(s) = r$.

(*Hint: Prove that g is unbounded, strictly increasing and use the Intermediate Value theorem judiciously.*)

9. Define a function $f : [2, \infty) \to \mathbf{R}$ by $f(x) = \displaystyle\int_2^x \frac{1}{t\sqrt{t^2 - 1}}\, dt.$

a) State your favourite version of the Mean Value Theorem *for derivatives.*

b) Prove that f is strictly increasing on $[2, \infty)$.

c) *Briefly* explain why f is differentiable on $(2, \infty)$, and find $f'(x)$. (Use theorems!)

Now denote $f(3)$ by q and let $g : [0, q] \to [2, 3]$ be the inverse function for f, which we know exists by parts (b) and (c). Define $h : [0, q] \to \mathbf{R}$ by

$$h(x) = \sqrt{g^2(x) - 1}, \quad \forall x \in [0, q].$$

d) *Briefly* explain why g and h are differentiable on $(0, q)$, and show that

$$g'(x) = g(x)h(x), \quad \forall x \in (0, q),$$

and

$$h'(x) = g^2(x), \quad \forall x \in (0, q).$$

(Use theorems!)

10. (Bonus) Suppose $A \subseteq \mathbf{R}$ and that suppose $f : A \to \mathbf{R}$ is uniformly continuous on A.

Remark. You may assume that if $\{a_n\}_{n\geq 1} \subseteq A$ is Cauchy, then so is $\{f(a_n)\}_{n\geq 1}$.

a) Suppose $c \notin A$, and that $\{a_n\}_{n\geq 1} \subseteq A$ and $\{b_n\}_{n\geq 1} \subseteq A$ are two sequences with $a_n \to c$ and $b_n \to c$. Prove that $\{f(a_n)\}_{n\geq 1}$ and $\{f(b_n)\}_{n\geq 1}$ must have the same limit.

b) Suppose $c \in \mathbf{R}$ is a limit point of A but $c \notin A$. Let $B = A \cup \{c\}$. Prove that there is a unique (well-defined!) and *continuous* function $g : B \to \mathbf{R}$ such that $\forall a \in A$, $g(a) = f(a)$.

(*Hint: Use the remark above to define $g(c)$, and use (a) to check that your definition doesn't depend on any choices you make. Then show that g is continuous on B!*)