uOttawa University of Ottawa

Faculty of Science »
Department of mathematics and statistics

MAT 2125 Mid-Term Examination 2013

February 28, 2013. Duration: 80 minutes

Instructor: Barry Jessup

Family Name:______________________

First Name:________________________

Student number:___________________

1	
2	
3	
4	
5	
(Bonus) 6	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY

1. The correct answer requires reasonable justification written legibly and logically. Proofs and explanations must be clear. Use full and grammatically correct mathematical sentences. Unless otherwise stated, you may use known theorems, but be sure to verify their hypotheses, and wherever possible name the theorems. You must convince me that you know why your solution is correct.
2. Questions 1-5 are worth an equal number of points. Question 6 is a bonus question (so that the maximum on the test is 110%), and **should not be attempted until all parts of questions 1-5 have been completed and checked.** It is much more difficult to earn points in the bonus question.
3. Please use the space provided, including the backs of pages if necessary. If you need scrap paper, please ask.
4. You have 80 minutes to complete this exam. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted.
5. Good luck, bonne chance!

1. a) If $\{a_n\}_{n\geq 1}$ is a real sequence, and $a \in \mathbf{R}$, give the definition of

$$\lim_{n\to\infty} a_n = a.$$

Now define a sequence $\{a_n\}_{n\geq 1}$ recursively by

$$a_{n+1} = \begin{cases} 0, & \text{if } n = 0 \\ \sqrt{a_n + 6}, & \text{if } n \geq 1. \end{cases}$$

You may assume that $0 \leq a_n < 3$, for all $n \geq 1$. In part (b), you may use the fact that $x > y \Rightarrow \sqrt{x} > \sqrt{y}$, and in part (c) you may use theorems about limits to find the limit, as well as the fact that $x \mapsto \sqrt{x}$ is continuous on $[0, \infty)$.

b) Prove that $a_n < a_{n+1}$, for all $n \geq 0$. (*Hint: use induction.*)

c) Prove (using a theorem) that $\{a_n\}_{n\geq 1}$ converges, and find its limit.

2. a) Define what is meant by "$\{a_n\}_{n\geq 1}$ is a Cauchy sequence".

Define a sequence $\{b_n\}_{n\geq 1}$ by

$$b_n = \begin{cases} 20 + \frac{1}{n}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

b) Find $\limsup_{n\to\infty} b_n$ and $\liminf_{n\to\infty} b_n$.

c) Use (b) and a theorem to prove that $\{b_n\}_{n\geq 1}$ is *not* Cauchy.

d) If $n_k = k^2 + k$, $\forall k \geq 1$, does the subsequence $\{b_{n_k}\}_{k\geq 1}$ of $\{b_n\}_{n\geq 1}$ converge? If so, give its limit. If not, explain why.

3. a) Let $\{c_n\}_{n\geq 1}$ be a real sequence. Define

$$\text{“The series } \sum_{n\geq 1} c_n \text{ converges.”}$$

Now consider the series

$$\sum_{n\geq 1}(-1)^n \frac{n}{n^2+1}$$

b) Does this series converge?

c) Does this series converge absolutely?

(In (b) and (c), you may known theorems, but be sure to verify their hypotheses.)

4. Let A and B be two subsets of $\mathbf{R}^2$.

a) Define "A is closed".

b) Define "B is compact".

c) State a theorem giving necessary and sufficient conditions for B to be compact. (*Do not* simply repeat the definition in (c).)

d) Now suppose A is closed and B is compact. Prove that

$$A \cap B = \{v \in \mathbf{R}^2 \mid v \in A \textit{ and } v \in B\}$$

is also compact.

5. Let $A \subseteq \mathbf{R}$, $a \in A$ and $f : A \to \mathbf{R}$.

a) Define

$$\text{“ The function } f \text{ is continuous at a.”}$$

b) Prove that $\forall x \in \mathbf{R}, \quad |x-1| < 1 \Longrightarrow |x-3| > 1.$

c) Define $f : \mathbf{R} \setminus \{3\} \to \mathbf{R}$ by $f(x) = \dfrac{2}{x-3}$. Prove carefully that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$x \in \mathbf{R} \setminus \{3\} \text{ and } |x-1| < \delta \quad \Longrightarrow \quad |f(x)+1| < \varepsilon.$$

(Hint: Part (b) may be useful.)

6. (Bonus) Suppose both series $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.