MAT 2125 W13 Mid Term

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2 **1.** a) If $\{a_n\}_{n\geq 1}$ is a real sequence, and $a \in \mathbf{R}$, give the definition of

$$\lim_{n\to\infty} a_n = a.$$

Now define a sequence $\{a_n\}_{n\geq 1}$ recursively by

$$a_{n+1} = \begin{cases} 0, & \text{if } n = 0 \\ \sqrt{a_n + 6}, & \text{if } n \geq 1. \end{cases}$$

You may assume that $0 \leq a_n < 3$, for all $n \geq 1$.

2 b) Prove that $a_n < a_{n+1}$, for all $n \geq 0$. (*Hint: use induction.*)

3 c) Prove (using a theorem) that $\{a_n\}_{n\geq 1}$ converges, and find its limit. (You may use theorems about limits to find the limit, as well as the fact that $s : [0,\infty) \to \mathbf{R}$ defined by $s(x) = \sqrt{x}$ is continuous on $[0,\infty)$).

2 a) $\lim_{n\to\infty} a_n = a \iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \geq 1$, $|a - a_n| < \varepsilon$

2 b) let $P(n)$ be the statement $a_n < a_{n+1}$. Since $a_0 = 0 < \sqrt{6} = a_1$, $P(0)$ holds. Now assume $a_n < a_{n+1}$. Then $a_{n+2} = \sqrt{a_{n+1}+6} > \sqrt{a_n + 6} = a_{n+1}$,

so $P(n) \Rightarrow P(n+1)$. Hence, $\forall n \geq 0$, $P(n)$ holds.

3 c) Since $a_n \uparrow$ and $\{a_n\}_{n\geq 1}$ is bdd above by 3, by Bolzano's thm, $\lim_{n\to\infty} a_n = \sup\{a_n | n \geq 1\}$ exists. 2

Since $a_{n+1} = \sqrt{a_n + 6}$ $\forall n \geq 1$,

$$a := \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n + 6} = \sqrt{\lim_{n\to\infty}(a_n + 6)}$$

$$= \sqrt{\lim_{n\to\infty} a_n + 6}$$

$$= \sqrt{a+6}$$

Hence $a^2 = a + 6$ or $a^2 - a - 6 = 0$. Thus

$$a = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2};$$ Hence $a = 3$ or -2.

Since $a_n \in [0,3)$ $\forall n$, $a = 3$. 1

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2. a) Define what is meant by "$\{a_n\}_{n\geq 1}$ is a Cauchy sequence".

Define a sequence $\{b_n\}_{n\geq 1}$ by

$$b_n = \begin{cases} 20 + \frac{1}{n}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

b) Find $\limsup\limits_{n\to\infty} b_n$ and $\liminf\limits_{n\to\infty} b_n$.

c) Use (b) and a theorem to prove that $\{b_n\}_{n\geq 1}$ is *not* Cauchy.

d) If $n_k = k^2 + k$, does the subsequence $\{b_{n_k}\}_{k\geq 1}$ of $\{b_n\}_{n\geq 1}$ converge? If so, give its limit. If not, explain why.

(2) a) $\{a_n\}_{n\geq 1}$ is Cauchy if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ st $\forall n, p \geq N$, $|a_n - a_p| < \varepsilon$

(3) b) By defn $\limsup\limits_{n\to\infty} b_n = \underbrace{\lim_{n\to\infty} \sup\{b_k \mid k \geq n\}}_{1} = \lim\limits_{n\to\infty} \sup\{20 + \frac{1}{2k+1} \mid 2k+1 \geq n\}$

$= \lim\limits_{n\to\infty} (20 + \frac{1}{n})$ 1/2

$= 20.$

Similarly, $\liminf\limits_{n\to\infty} b_n = \underbrace{\lim_{n\to\infty} \inf\{b_k \mid k \geq n\}}_{1} = \lim\limits_{n\to\infty} 0 = 0$ 1/2

(1) c) But $\{b_n\}_{n\geq 1}$ is Cauchy $\Leftrightarrow$ $\{b_n\}_{n\geq 1}$ converges (Cauchy)

$\Leftrightarrow \limsup\limits_{n\to\infty} b_n = \liminf\limits_{n\to\infty} b_n$ (Cauchy),

and we know $\limsup\limits_{n\to\infty} b_n = 20 > 0 = \liminf\limits_{n\to\infty} b_n$, hence

$\{b_n\}_{n\geq 1}$ is not Cauchy

(1) d) Since $k^2 + k = k(k+1)$ is always even,

$b_{n_k} = 0$, and so $\lim\limits_{n\to\infty} b_{n_k} = 0.$

3. a) Let $\{c_n\}_{n\geq 1}$ be a real sequence. Define

"The series $\sum_{n\geq 1} c_n$ converges."

Now consider the series

$$\sum_{n\geq 1}(-1)^n \frac{n}{n^2+1}$$

b) Does this series converge?

c) Does this series converge absolutely?

(In (b) and (c), You may known theorems, but be sure to verify their hypotheses.)

(2) a) The series $\sum_{n\geq 1} c_n$ converges iff the sequence $\left\{ \sum_{k=1}^{n} c_n \right\}_{n\geq 1}$ of its partial sums converge.

(3) b) This is an alternating series. Now $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}}$ $= 0$ (1). Moreover, $\frac{n+1}{(n+1)^2+1} \leq \frac{n}{n^2+1} \iff (n+1)(n^2+1) \leq n\{(n+1)^2+1\}$

$\iff n^3+n+n^2+1 \leq n(n^2+2n+1)+n \iff \cancel{n^3}+\cancel{n}+\cancel{n^2}+1 \leq \cancel{n^3}+\cancel{2}n^2+\cancel{2n}$

$\iff 1 \leq n$, the sequence $\frac{n}{n^2+1}$ is decreasing (1)

Hence, by Leibniz, the alternating series $\sum_{n\geq 1} (-1)^n \frac{n}{n^2+1}$ converges (1)

(2) c) Since $\frac{n}{n^2+1} \geq \frac{1}{2n}$ (1) $\iff 2n^2 \geq n^2+1 \iff n^2 \geq 1$, which holds $\forall n \geq 1$, $\sum_{n\geq 1} \frac{n}{n^2+1}$ diverges, by comparison with $\sum_{n\geq 1} \frac{1}{2n} = \frac{1}{2}\sum_{n\geq 1}\frac{1}{n}$, which we know diverges (1)

4. Let A and B be two subsets of $\mathbf{R}^2$.

(1) a) Define "A is closed".

~~b) State a theorem giving necessary and sufficient conditions for A to be closed, in terms of sequences $\{a_n\}_{n\geq 1} \subseteq A$.~~

(2) b) Define "B is compact".

(2) c) State a theorem giving necessary and sufficient conditions for B to be compact. (*Do not* simply repeat the definition in (c).)

(2) d) Now suppose A is closed and B is compact. Prove that

$$A \cap B = \{v \in \mathbf{R}^2 \mid v \in A \textit{ and } v \in B\}$$

is also compact.

a) The subset A is closed if A^c is open (1)

~~b)~~ The subset A is closed $\Leftrightarrow$ $\forall$ convergent sequences $\{a_n\}_{n\geq 1} \subseteq A$, if $a_n \to a$, then $a \in A$.

b) B is compact if every sequence in B has a convergent subsequence (1) whose limit lies in B. (1)

c) B is compact iff B is closed (1) and bounded. (1)

(2) d) Let $\{a_n\}_{n\geq 1} \subseteq A \cap B$. Since $\{a_n\}_{n\geq 1} \subseteq B$ and B is compact, $\exists$ convergent subsequence $\{a_{n_k}\}_{k\geq 1}$ s.t. $\lim_{k\to\infty} a_{n_k} = a \in B$.

Since the convergent sequence $\{a_{n_k}\}_{k\geq 1} \subseteq A$, and A is closed, then $a \in A$. Hence $a \in A \cap B$. Thus the sequence $\{a_n\}_{n\geq 1}$ has a convergent subsequence with limit in $A \cap B$. Hence $A \cap B$ is compact, by defn.

OR: Since $\exists D > 0$ st $\forall b \in B$, $|b| \leq D$, $\forall c \in A \cap B$, $|c| \leq D$, so $A \cap B$ is bounded. Moreover, the intersection of 2 closed sets is closed (DeM) and so $A \cap B$ is closed & bdd. Thus by (d), $A \cap B$ is compact.

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5. Let $A \subseteq \mathbf{R}$, $a \in A$ and $f : A \to \mathbf{R}$.

(2) a) Define

"The function f is continuous at a."

(1) b) Prove that $\forall x \in \mathbf{R}$, $|x-1| < 1 \Longrightarrow |x-3| > 1$.

Define $f : \mathbf{R} \setminus \{3\} \to \mathbf{R}$ by

$$\forall \varepsilon > 0 \qquad f(x) = \frac{2}{x-3}.$$

c) ~~For any $\varepsilon > 0$~~, prove carefully that, f there exists a $\delta > 0$ such that

$$x \in \mathbb{R}\setminus\{3\} \text{ and } |x-1| < \delta \Longrightarrow |f(x) + 1| < \varepsilon.$$

(2) a) The fn f is cts at a if $\forall \varepsilon > 0 \, \exists \delta > 0$ st $x \in A$ and $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

(1) b) Note that $2 = |x-3 - x+1| \leq |x-3| + |x-1|$, so

$|x-3| \geq 2 - |x-1| > 2-1$ if $|x-1| < 1$. Thus $|x-1| < 1 \Rightarrow |x-3| < 1$.

OR $x-1 < 1 \Leftrightarrow 0 < x < 2 \Rightarrow -3 < x-3 < -1 \Rightarrow |x-3| > 1$.

(4) c) Note that $|f(x) + 1| = \left| \frac{2}{x-3} + 1 \right| = \left| \frac{2+x-3}{x-3} \right| = \frac{|x-1|}{|x-3|}$.

(1)

Let $\varepsilon > 0$ and let $\delta = \min\{1, \varepsilon\}$. Then as $|x-1| < \delta$

$\Rightarrow |x-1| < 1 \Rightarrow |x-3| > 1 \Rightarrow \frac{1}{|x-1|} < 1 \Rightarrow \frac{|x-1|}{|x-3|} < |x-1| < \varepsilon$,

Hence $|x-1| < \delta \Rightarrow |f(x) - f(1)| < \varepsilon$.

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(1) - any correct δ

(2) proof that δ works

$3\frac{1}{2}$ **6.** (Bonus) Suppose both series $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Note that $(|a_n| - |b_n|)^2 \geq 0$

$$\Rightarrow \quad |a_n|^2 + |b_n|^2 - 2|a_n b_n| \geq 0$$

$$\Rightarrow \quad \tfrac{1}{2}(a_n^2 + b_n^2) \geq |a_n b_n| \geq 0 \quad , \quad \forall n \geq 1$$

Hence, the partial sums of $\sum_{n\geq 1} |a_n b_n|$ are bounded above by $1/2$ the sum of the partial sums of $\sum_{n\geq 1} a_n^2$ and $\sum_{n\geq 1} b_n^2$. Thus, by comparison, $\sum_{n\geq 1} |a_n b_n|$ converges, and so too does $\sum_{n\geq 1} a_n b_n$.