uOttawa University of Ottawa

Faculty of Science »
Department of mathematics and statistics

MAT 2125 Final Examination 2013

26 April, 2013. Duration: 3 hours

Instructor: Barry Jessup

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(Bonus) 10	
Total	

Family Name:______________________

First Name:______________________

Student number:____________________

PLEASE READ THESE INSTRUCTIONS CAREFULLY

1. The correct answer requires reasonable justification written legibly and logically. Proofs and explanations must be clear. Use full and grammatically correct mathematical sentences. Unless otherwise stated, you may use known theorems, but be sure to verify their hypotheses, and wherever possible name the theorems. You must convince me that you know why your solution is correct.
2. Questions 1-9 are worth an equal number of points. Question 10 is a bonus question (so that the maximum on the test is 110%), and **should not be attempted until all parts of questions 1-9 have been completed and checked.** It is much more difficult to earn points in the bonus question.
3. Please use the space provided, including the backs of pages if necessary. If you need scrap paper, please ask.
4. You have 3 hours minutes to complete this exam. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted.
5. Good luck, bonne chance!

1. Let $A \subseteq \mathbf{R}$.

a) If $s \in \mathbf{R}$, define what is meant by "s is the supremum of A", i.e. $s = \sup A$.

b) If $l \in \mathbf{R}$, define what is meant by "l is the infimum of A", i.e. $l = \inf A$.

c) State necessary and sufficient conditions for $\sup A$ to exist.

Now suppose both $l = \inf A$, and $s = \sup A$ exist.

d) Prove that $s - l = \sup\{\, a - a' \;\mid\; a,\ a' \in A\}$.

2. Let $\{a_n\}_{n\geq 1}$ be a real sequence.

a) Define "$\{a_n\}_{n\geq 1}$ is a Cauchy sequence."

b) Define "$\{a_n\}_{n\geq 1}$ is a bounded sequence."

c) State the triangle inequality for $\mathbf{R}$.

d) Prove that a Cauchy sequence is bounded, *directly from the definitions in (a), (b), and (c).*

e) Give an example of a Cauchy sequence $\{a_n\}_{n\geq 1}$ with $a_n > 0$, $\forall n \geq 1$, for which $\left\{\frac{1}{a_n}\right\}_{n\geq 1}$ is *not* Cauchy. (You may use theorems from class.)

3. a) Let $\{c_n\}_{n\geq 1}$ be a real sequence. Define

$$\text{“The series } \sum_{n=1}^{\infty} c_n \text{ converges.”}$$

Now consider the two series

$$A: \quad \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$B: \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n^2+1}}$$

b) Does the series A converge?

c) Does the series B converge?

d) Does the series B converge absolutely?

(In (b)–(d), you may use known tests and theorems, but be sure to verify their hypotheses.)

4. Let $A \subseteq \mathbf{R}$, $a \in A$ and $f : A \to \mathbf{R}$.

a) Define

$$\text{“ The function } f \text{ is continuous at a.”}$$

b) Prove that $\forall x \in \mathbf{R}, \quad |x| < \frac{1}{2} \Longrightarrow |1 - x^2| > \frac{3}{4}$.

c) Define $f : \mathbf{R} \setminus \{-1, 1\} \to \mathbf{R}$ by $f(x) = \dfrac{1}{1 - x^2}$.

Prove carefully that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$x \in \mathbf{R} \setminus \{-1, 1\} \text{ and } |x| < \delta \quad \Longrightarrow \quad |f(x) - 1| < \varepsilon.$$

5. Let A and B be subsets of $\mathbf{R}^2$. Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous on $\mathbf{R}^2$ and that $U \subseteq \mathbf{R}$ is an open set.

a) Define "A is open".

b) Define "B is compact".

c) State a *sufficient* condition for A to be open, in terms of f and U.

Now define $C = \{(x, y) \in \mathbf{R}^2 \mid xy > 0\}$, and $D = \{(x, y) \in \mathbf{R}^2 \mid xy = 0\}$.

d) Prove that C is open.

e) Prove that D is closed, but is not compact.

6. Let $A \subset \mathbf{R}^n$, $b \in \mathbf{R}^n$, and suppose $f : A \to \mathbf{R}^p$ is uniformly continuous on A.

a) Define "b is a limit point of A", *without referring to sequences in* A.

b) Give a characterization of limit points in terms of sequences $\{a_n\}_{n\geq 1} \subseteq A$.

c) Define "f is uniformly continuous on A."

d) Prove that if $\{a_n\}_{n\geq 1} \subseteq A$ and $a_n \to b \in \mathbf{R}^n$, then $\{f(a_n)\}_{n\geq 1}$ is Cauchy. (Note that you may *not* assume that f is defined at b.)

7. Define $f : [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, \frac{1}{2}] \\ 1 & \text{for } x \in (\frac{1}{2}, 1] \end{cases}$$

a) Carefully state a necessary and sufficient condition for f to be Riemann (-Darboux) integrable in terms of upper sums $U(f,P)$ and lowers sums $L(f,P)$, where P denotes a partition of $[0,1]$. (You may give the definition or an equivalent condition.)

b) For $n \in \mathbf{N}, n \geq 1$, let P_n be the partition $P_n = \{0, \frac{1}{2}, \frac{1}{2} + \frac{1}{n}, 1\}$. Find $U(f, P_n)$ and $L(f, P_n)$, for all $n \geq 1$.

c) Use your result in (b) and your response in (a) to directly prove that f is integrable.

8. Define a function $f : [0, \frac{1}{2}] \to \mathbf{R}$ by by $f(x) = \displaystyle\int_0^x \frac{1}{\sqrt{1-t^2}}\, dt$.

a) *Briefly* explain why f is continuous on $[0, \frac{1}{2}]$ and differentiable on $(0, \frac{1}{2})$. (Use theorems!)

b) State your favourite version of the Mean Value Theorem *for derivatives*.

c) Prove that f is strictly increasing on $[0, \frac{1}{2}]$.

Now denote $f(\frac{1}{2}) = p$. Let

$$s : [0, p] \to [0, \frac{1}{2}]$$

be the inverse function for f, which we know exists by parts (a) and (c), and define $c : [0, p] \to \mathbf{R}$ by

$$c(x) = \sqrt{1 - s^2(x)}, \quad \forall x \in [0, p].$$

d) *Briefly* explain why s and c are differentiable on $(0, p)$, and show that $s'(x) = c(x)$, and $c'(x) = -s(x)$, for all $x \in (0, p)$. (Use theorems!)

9. a) Suppose $f_n : [a, b] \to \mathbf{R}, n \geq 1$ is a sequence of functions, and $f : [a, b] \to \mathbf{R}$ is a function. Define what is meant by

$$\sum_{n=1}^{\infty} f_n \quad \textit{converges uniformly to } f \text{ on } [a, b].$$

b) Define $h : \mathbf{R} \setminus \{1\} \to \mathbf{R}$ by $h(x) = \dfrac{1}{1 + x}$.

(i) Show that $h^{(n)}(0) = (-1)^n n!$, for all $n \in \mathbf{N}$.

(ii) Give the Taylor polynomial of h of order 3 at 0, <u>and the remainder term</u>.

(iii) Give the Taylor series for h at 0 and show it converges uniformly on any closed interval $[-r, r]$, for $r < 1$.

(iv) Prove that the Taylor series for h at 0 does not converge to h at $x = 1$, even though $h(1) = \frac{1}{2}$.

(v) Prove that the Taylor series for h at 0 converges to h on $(-1, 1)$.

10. (Bonus) Define a function $g : [0, 1] \to \mathbf{R}$ by

$$g(x) = \begin{cases} 0, & \text{if } x = 0 \text{ or } x \text{ is irrational,} \\ \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \text{ with } \gcd(p, q) = 1. \end{cases}$$

Prove that g is integrable and $\int_0^1 g = 0$