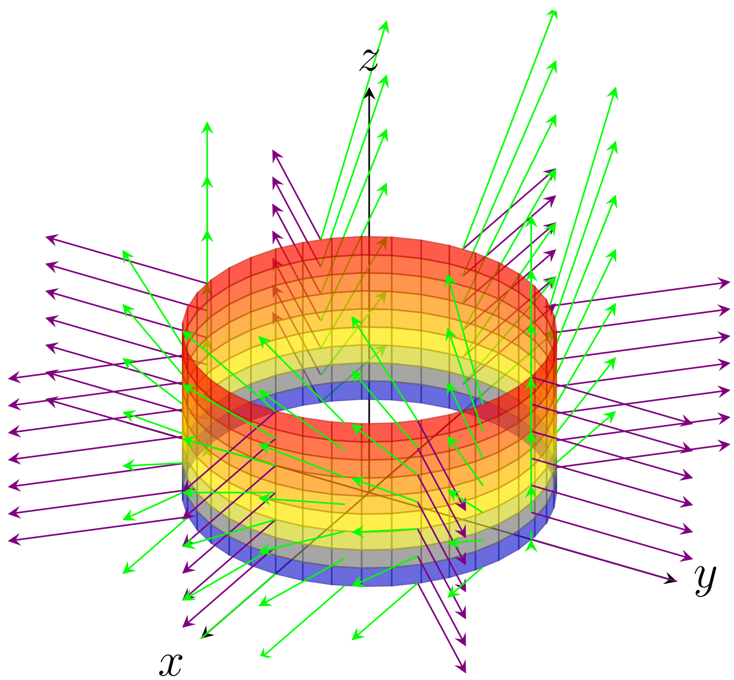
Multivariable Calculus

MAT 2122

Fall 2021



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Preface

These are notes for the course *Multivariable calculus* (MAT 2122) at the University of Ottawa. This is a course on calculus in multiple dimensions aimed at students majoring in mathematics or doing a joint degree with another subject. We will focus mainly on real-valued functions of multiple real-valued inputs. Many concepts will be discussed using the language of vectors and linear algebra, since this is the most natural setting for multivariable calculus. We'll see how much of the calculus you learned in previous courses generalizes to multiple dimensions. This will allow us to explore interesting new mathematics, such as integration over surfaces and three-dimensional regions.

Acknowledgements: I would like to thank Aaron Tikuisis and Tanya Schmah for sharing with me their lecture notes for this course, upon which portions of these lecture notes are based. The official textbooks for this course are the open access books [FRYc, FRYe]. Portions of these notes follow those textbooks.

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Course website: `https://alistairsavage.ca/mat2122`

Conventions

We will use the following conventions in these notes:

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ denotes the set of nonnegative integers.
- $\mathbb{R}$ denotes the set of real numbers.
- We will denote vectors by bold letters, such as $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{w}$. Since bold characters can be hard to handwrite (e.g. during lectures), we will sometimes denote vectors with a small overhead arrow, e.g. $\vec{x}, \vec{y}, \vec{u}, \vec{v}, \vec{w}$, when writing by hand.

We will write vectors in $\mathbb{R}^n$ horizontally as

$$\mathbf{x} = (x_1, \dots, x_n) \tag{1}$$

or as column matrices

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \tag{2}$$

interchangeably. Note that (1) is *not* the same as the row matrix

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}.$$

Rather, (1) is just a horizontally written notation (to save space) for the column matrix (2). We write

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

for the norm of a vector. Note that some textbooks, including [FRYe], use the notation $|\mathbf{x}|$ for this norm.

Exercises

After most sections, there is a list of exercises. Some exercises are written directly in the notes, but most refer to the problems book of the course textbooks. Sometimes the list of exercises is quite long, but many of the problems are similar. In this case, you should attempt a selection of them—enough so that you feel comfortable doing that type of question. *Doing exercises is the best way to learn the material!*

Chapter 1

Differentiation

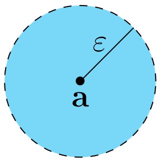
In this first chapter, we discuss differentiation in multiple dimensions. Some results will be review from previous courses, while other material will be new.

1.1 Open sets and boundaries

Many definitions in calculus require one to be able to get "near" a point. The precise formulation of this concept involves the notion of open sets, which we discuss here.

Definition 1.1.1 (Open ball)**.** Let $\mathbf{a} \in \mathbb{R}^n$ and $\varepsilon > 0$. The *open ball* of radius ε centered at $\mathbf{a}$ is

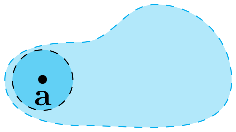
$$B_\varepsilon(\mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < \varepsilon\}.$$



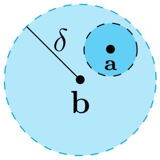
Definition 1.1.2 (Open set)**.** A set $A \subseteq \mathbb{R}^n$ is *open* if

$$\forall \mathbf{a} \in A,\ \exists \varepsilon > 0 \text{ such that } B_\varepsilon(\mathbf{a}) \subseteq A.$$

In other words, A is open if every point of A is the center of an open ball contained in A.



Example 1.1.3. Every open ball is an open set. Let's prove it! Choose $\mathbf{b} \in \mathbb{R}^n$ and $\delta > 0$. We will prove that $A = B_\delta(\mathbf{b})$ is open. Let $\mathbf{a} \in A$. We must find $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{a}) \subseteq A$.



Since $\mathbf{a} \in A = B_\delta(\mathbf{b})$, we have $\|\mathbf{a} - \mathbf{b}\| < \delta$. Let

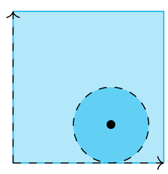
$$\varepsilon = \delta - \|\mathbf{a} - \mathbf{b}\| > 0.$$

Then, for all $\mathbf{x} \in B_\varepsilon(\mathbf{a})$, we have $\|\mathbf{x} - \mathbf{a}\| < \varepsilon$, and so

$$\begin{aligned}
\|\mathbf{x} - \mathbf{b}\| &\le \|\mathbf{x} - \mathbf{a}\| + \|\mathbf{a} - \mathbf{b}\| && \text{(triangle inequality)} \\
&< \varepsilon + \|\mathbf{b} - \mathbf{a}\| \\
&= \delta - \|\mathbf{a} - \mathbf{b}\| + \|\mathbf{b} - \mathbf{a}\| \\
&= \delta.
\end{aligned}$$

Thus $x \in B_\delta(\mathbf{b}) = A$. Hence $B_\varepsilon(\mathbf{a}) \subseteq A$, as desired. △

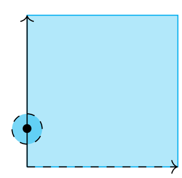
Example 1.1.4. The set $A = (0, \infty) \times (0, \infty) \subseteq \mathbb{R}^2$ is open. Indeed, if $(a, b) \in A$, then $a, b > 0$. If we take $\varepsilon = \min\{a, b\}$, then $B_\varepsilon(a, b) \subseteq A$.



△

Example 1.1.5. The set $A = [0, \infty) \times (0, \infty)$ is *not* open. To see this, we note that for any $b > 0$, we have $(0, b) \in A$. However, for *all* $\varepsilon > 0$, we have

$$\left(-\tfrac{\varepsilon}{2}, b\right) \in B_\varepsilon(0, b) \quad \text{but} \quad \left(-\tfrac{\varepsilon}{2}, b\right) \notin A.$$



Hence $B_\varepsilon(0, b) \nsubseteq A$. △

Recall that, if A and B are sets, then their *set difference* is

$$A \setminus B := \{x : x \in A,\ x \notin B\}.$$

Definition 1.1.6 (Boundary)**.** Let $A \subseteq \mathbb{R}^n$. The *boundary* of A is the set of all points $\mathbf{a} \in \mathbb{R}^n$ such that,

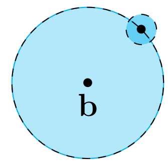
$$\forall \varepsilon > 0 \ \ (B_\varepsilon(\mathbf{a}) \cap A \neq \varnothing \text{ and } B_\varepsilon(\mathbf{a}) \setminus A \neq \varnothing.)$$

We denote the boundary of A by ∂A.

Note that Definition 1.1.6 does *not* require that $\mathbf{a} \in A$.

Remark 1.1.7. In Section 8.4 we will give a different definition of boundary for surfaces in $\mathbb{R}^3$.

Example 1.1.8. If $A = B_\delta(\mathbf{b}) \subseteq \mathbb{R}^n$, then $\partial A = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{b}\| = \delta\}$.



In this case, *none* of the boundary points of A are in A. The *closed* ball has the same boundary:

$$\partial\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{b}\| \leq \delta\} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{b}\| = \delta\}.$$

In this case, *all* of the boundary points are in the set itself. △

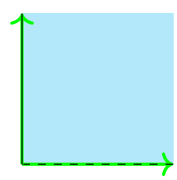
Example 1.1.9. If

$$A = [0, \infty) \times (0, \infty)$$

then

$$\partial A = \{(x, 0) : x \geq 0\} \cup \{(0, y) : y \geq 0\}.$$

The boundary is the region colored green in the following picture:



Note that the sets

$$(0, \infty) \times (0, \infty), \quad (0, \infty) \times [0, \infty), \quad \text{and} \quad [0, \infty) \times [0, \infty)$$

all have the same boundary as A. △

Exercises.

1.1.1. For each of the following sets, state whether or not the set is open and describe its boundary.

(a) $[0, 1] \times (0, 1)$

(b) $(0, 1) \times (0, 1)$

(c) $[0, 1] \times [0, 1]$

(d) $\{(x, y) : y \geq x^2\}$

(e) $\{(x, y) : y > x^2\}$

(f) $\mathbb{R}^2$

(g) $\varnothing$

1.2 Limits

The notion of a *limit* underpins all of calculus. In this section we give a precise definition of a limit in the multivariable setting, and discuss several important properties of limits.

Definition 1.2.1 (Limit)**.** Suppose $A \subseteq \mathbb{R}^n$ is an open set, $f\colon A \to \mathbb{R}^m$, $\mathbf{a} \in A \cup \partial A$, and $\mathbf{L} \in \mathbb{R}^m$. Then we write

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L}$$

if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } (\mathbf{x} \in A,\ 0 < \|\mathbf{x} - \mathbf{a}\| < \delta) \implies \|f(\mathbf{x}) - \mathbf{L}\| < \varepsilon.$$

(Note that the condition $0 < \|\mathbf{x} - \mathbf{a}\|$ is equivalent to the condition $\mathbf{x} \neq \mathbf{a}$.)

Remarks 1.2.2.

(a) As in single variable calculus, limits may not exist. However, if they exist, they are unique.

(b) Note that Definition 1.2.1 allows $\mathbf{a}$ to lie in the boundary of A, and does not require $\mathbf{a} \in A$. Thus, in some situations, we can discuss the limit of a function at a point that does not lie in the domain of that function.

In practice, it is enough to consider the $m = 1$ case of Definition 1.2.1, as we now explain. Suppose $A \subseteq \mathbb{R}^n$. Then every function $f\colon A \to \mathbb{R}^m$ can be written in the form

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x})), \quad \text{where} \quad f_i\colon A \to \mathbb{R} \text{ for } i = 1, 2, \dots, m. \tag{1.1}$$

The f_i are the *component functions* of f, and we write $f = (f_1, \dots, f_m)$. The following proposition states that the coordinates of the limit of f are limits of its component functions (if these all exist).

Proposition 1.2.3. *Suppose $A \subseteq \mathbb{R}^n$ is an open set, $f\colon A \to \mathbb{R}^m$, $\mathbf{a} \in A \cup \partial A$, and $\mathbf{L} = (L_1, \dots, L_m) \in \mathbb{R}^m$. Then*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L}$$

if and only if

$$\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = L_i \quad \text{for } i \in \{1, 2, \dots, m\}.$$

In particular $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ *exists if and only if* $\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x})$ *exists for each* i.

Proof. Suppose $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L}$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$(\mathbf{x} \in A,\ 0 < \|\mathbf{x} - \mathbf{a}\| < \delta) \implies \|f(\mathbf{x}) - \mathbf{L}\| < \varepsilon.$$

Let $i \in \{1, 2, \dots, m\}$. Since

$$|f_i(\mathbf{x}) - L_i| \le \sqrt{\sum_{j=1}^m (f_j(\mathbf{x}) - L_j)^2} = \|f(\mathbf{x}) - \mathbf{L}\|,$$

we have

$$(\mathbf{x} \in A,\ 0 < \|\mathbf{x} - \mathbf{a}\| < \delta) \implies |f_i(\mathbf{x}) - L_i| \le \|f(\mathbf{x}) - \mathbf{L}\| < \varepsilon.$$

Hence $\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = L_i$.

Conversely, suppose $\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = L_i$ for $i \in \{1,2,\dots,m\}$. Let $\varepsilon > 0$. For each $i \in \{1,2,\dots,m\}$, there exists $\delta_i > 0$ such that

$$(\mathbf{x} \in A,\ 0 < \|\mathbf{x}-\mathbf{a}\| < \delta_i) \implies |f_i(\mathbf{x}) - L_i| < \tfrac{\varepsilon}{\sqrt{m}}.$$

Let $\delta = \min\{\delta_1,\dots,\delta_m\}$. Then, for all $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x}-\mathbf{a}\| < \delta$, we have

$$\|f(\mathbf{x}) - \mathbf{L}\| = \sqrt{\sum_{j=1}^{m} (f_j(\mathbf{x}) - L_j)^2} \le \sqrt{\sum_{j=1}^{m} \tfrac{\varepsilon^2}{m}} = \varepsilon.$$

Hence $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L}$. □

Definition 1.2.4 (Continuity)**.** Suppose $A \subseteq \mathbb{R}^n$ is an open set, $f\colon A \to \mathbb{R}^m$ and $\mathbf{a} \in A$. We say f is *continuous at* $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

We say that f is *continuous* if it continuous at every point in its domain.

Corollary 1.2.5. *Suppose $A \subseteq \mathbb{R}^n$ is an open set, $f = (f_1,\dots,f_m)\colon A \to \mathbb{R}^m$ and $\mathbf{a} = (a_1,\dots,a_m) \in A$. Then f is continuous at $\mathbf{a}$ if and only if f_i is continuous at a_i for each $i \in \{1,2,\dots,m\}$.*

Proof. This follows from Definition 1.2.4 and Proposition 1.2.3. □

Remark 1.2.6. In light of Proposition 1.2.3 and Corollary 1.2.5, we will often restrict our attention to real-valued functions $f\colon A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$, when discussing limits and continuity. We will see later that we can also make this simplification when studying derivatives. However, we *cannot* make analogous simplifications in the *domain* of a function. For example, we will see that all the partial derivatives of a function may exist, even though the function is not differentiable.

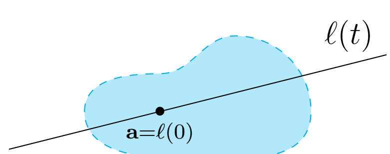
Theorem 1.2.7 (Squeeze theorem)**.** *Suppose $A \subseteq \mathbb{R}^n$ is open, $\mathbf{a} \in A \cup \partial A$, $L \in \mathbb{R}$, and $f, h\colon A \to \mathbb{R}$. If*

$$|f(\mathbf{x}) - L| \le h(\mathbf{x}) \text{ for all } \mathbf{x} \in A \text{ and } \lim_{\mathbf{x}\to\mathbf{a}} h(\mathbf{x}) = 0,$$

then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$.

Proof. The proof is very similar to the proof of the single-variable squeeze theorem, and so we will omit it. □

Proposition 1.2.8. *Suppose $f\colon A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is open, and $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$. Then, for any line $\ell\colon \mathbb{R} \to \mathbb{R}^n$ such that $\ell(0) = \mathbf{a}$, we have $\lim_{t\to 0} f(\ell(t)) = L$.*



Proof. The proof of this proposition is Exercise 1.2.1. □

Warning 1.2.9. The converse of Proposition 1.2.8 is false! There exist functions f such that $\lim_{t\to 0} f(\ell(t)) = L$ for all lines ℓ, but $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ does not exist. See [FRYc, §2.1.1] for an example.

Example 1.2.10. Let

$$f(x,y) = \frac{x^2 + 5xy + 3y^2}{|x| + |y|}.$$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If so, what is it? Let's try approaching $(0,0)$ along the line

$$\ell(t) = (t, 0).$$

We have

$$f(\ell(t)) = f(t,0) = \frac{t^2 + 5t\cdot 0 + 3\cdot 0^2}{|t| + |0|} = \frac{t^2}{|t|} = |t|.$$

Thus,

$$\lim_{t\to 0} f(\ell(t)) = \lim_{t\to 0} |t| = 0.$$

Therefore, by Proposition 1.2.8, *if* $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, it must be 0. Let's prove $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Using the triangle inequality, we have

$$|f(x,y) - 0| = \left|\frac{x^2 + 5xy + 3y^2}{|x| + |y|}\right| \le \frac{|x^2|}{|x| + |y|} + \frac{5|xy|}{|x| + |y|} + \frac{3|y^2|}{|x| + |y|}.$$

Now

$$\begin{aligned}
\frac{|x^2|}{|x| + |y|} = \frac{|x|\,|x|}{|x| + |y|} &\le \frac{|x|(|x| + |y|)}{|x| + |y|} = |x|,\\
\frac{5|xy|}{|x| + |y|} &\le \frac{5|x|(|x| + |y|)}{|x| + |y|} = 5|x|,\\
\frac{3|y^2|}{|x| + |y|} &\le \frac{3|y|(|x| + |y|)}{|x| + |y|} = 3|y|.
\end{aligned}$$

Therefore

$$|f(x,y) - 0| \le |x| + 5|x| + 3|y| = 6|x| + 3|y|.$$

Since $\lim_{(x,y)\to(0,0)}(6|x| + 3|y|) = 0$, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

by the squeeze theorem (Theorem 1.2.7). △

Example 1.2.11. Consider the function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If so, what is it? Let's try approaching the origin along the two coordinate axes. First consider $\ell_1(t) = (t,0)$. Then

$$f(\ell_1(t)) = f(t,0) = \frac{t^2 - 0^2}{t^2 + 0^2} = 1.$$

Now consider $\ell_2(t) = (0,t)$. Then

$$f(\ell_2(t)) = f(0,t) = \frac{0^2 - t^2}{0^2 + t^2} = -1.$$

Thus, by Proposition 1.2.8, $\lim_{(x,y)\to(0,0)}$ does not exist. △

We conclude this section with some properties of limits. The proofs of these properties are analogous to the proofs in the single-variable case, so we omit them. For the remainder of this section, $A \subseteq \mathbb{R}^n$ is open, $f, g \colon A \to \mathbb{R}$, and $\mathbf{a} \in A \cup \partial A$.

Proposition 1.2.12 (Uniqueness of limits)**.** *If*

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = K \quad \textit{and} \quad \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L,$$

then $K = L$.

Proposition 1.2.13 (Arithmetic of limits)**.** *Suppose $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$, $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$ both exist.*

(a) $\lim_{\mathbf{x}\to\mathbf{a}}(f(\mathbf{x}) + g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})$.

(b) $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})g(\mathbf{x}) = \left(\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\right)\left(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\right)$.

(c) *If $f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in A$ and $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) \neq 0$, then*

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{1}{f(\mathbf{x})} = \frac{1}{\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})}.$$

Corollary 1.2.14. *If $\mathbf{a} \in A$ and f, g are both continuous at $\mathbf{a}$, then so are $f + g$, fg, and $\frac{1}{f}$ (if defined).*

Exercises.

1.2.1. Prove Proposition 1.2.8.

Exercises from [FRYb, §2.1]: Q1–Q11.

1.3 Affine functions

Suppose we have a function

$$f\colon \mathbb{R}^n \to \mathbb{R}^m, \qquad n, m \in \mathbb{N}.$$

Recall, from linear algebra, that f is *linear* if

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}), \qquad f(s\mathbf{x}) = sf(\mathbf{x}),$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $s \in \mathbb{R}$. The function f is linear if and only if it is given by multiplication by some matrix. In other words, f is linear if and only if there exists an $m \times n$ matrix A such that

$$f(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

where we view $\mathbf{x}$ as an $n \times 1$ column matrix.

More generally, the function f is *affine* if it is a composition of a linear map and a translation. In other words, f is affine if and only if there exists an $m \times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^m$ such that

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}.$$

Intuitively, both linear and affine functions are those functions whose graphs are “flat”. However, the graphs of linear functions are required to pass through the origin. Note that if $\mathbf{x}_0 \in \mathbb{R}^m$ is some fixed vector, then the function $\mathbf{x} \mapsto A(\mathbf{x} - \mathbf{x}_0) + \mathbf{b}$ is also affine, since we have

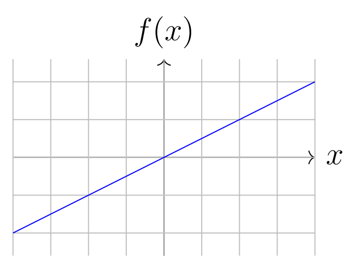
$$A(\mathbf{x} - \mathbf{x}_0) + \mathbf{b} = \underbrace{A\mathbf{x}}_{\text{linear}} + \underbrace{(\mathbf{b} - A\mathbf{x}_0)}_{\text{translation}}.$$

Many of the affine maps we are about to see will be of this form.

Example 1.3.1. The function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = \tfrac{1}{2}x,$$

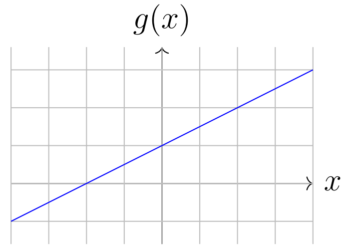
is linear. Its graph is as follows:



Note that the graph passes through the origin. On the other hand, the function

$$g\colon \mathbb{R} \to \mathbb{R}, \quad g(x) = \tfrac{1}{2}x + 1,$$

is *not* linear, but it *is* affine. Its graph is as follows:



Note that the graph does not pass through the origin. △

Exercises.

1.3.1. Which of the following functions are linear? Which are affine?

(a) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 2x - 7y + 10$

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 2(x - 1) + 5(y - 3) + 17$

(c) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 + y$

(d) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x + 4y, 3x + y, 1)$

(e) $f\colon \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (z, x)$

1.4 Affine approximations in one and two variables

Since linear and affine functions are much easier to work with than arbitrary functions (because we can use the tools of linear algebra), it is often useful to approximate a function near some point by an affine function. You have done this in previous calculus courses when you found *tangent lines* and *tangent planes*. A tangent line is the graph of the affine function that best approximates a given function $\mathbb{R} \to \mathbb{R}$, while a tangent plane is the graph of the affine function that best approximates a given function $\mathbb{R}^2 \to \mathbb{R}$. Note that it is important that we allow ourselves to work with *affine* functions, and not just *linear* ones, since the best approximation will not pass through the origin in general.

Recall that if $A \subseteq \mathbb{R}$ is open (e.g. A is some open interval (b, c) for $b < c$), and $f\colon A \to \mathbb{R}$, then the *derivative* of f at $a \in A$ is

$$f'(a) = \frac{\mathrm{d}f}{\mathrm{d}x}(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists. If this limit exists, we say f is *differentiable* at a. We say f is *differentiable* if it is differentiable at all $a \in A$.

Lemma 1.4.1. *The function f is differentiable at a with derivative $f'(a)$ if and only if*

$$\lim_{x\to a} \frac{f(x) - \big(f(a) + f'(a)(x-a)\big)}{|x-a|} = 0. \tag{1.2}$$

Proof. Suppose first that f is differentiable at a with derivative $f'(a)$. Then

$$\lim_{x\to a^+} \frac{f(x) - \big(f(a) + f'(a)(x-a)\big)}{|x-a|} = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} - f'(a) = f'(a) - f'(a) = 0$$

and

$$\lim_{x\to a^-} \frac{f(x) - \big(f(a) + f'(a)(x-a)\big)}{|x-a|} = \lim_{x\to a^-} \frac{f(x)-f(a)}{a-x} + f'(a) = -f'(a) + f'(a) = 0.$$

Since the two one-sided limits exist and are equal, we have proved (1.2). The proof of the reverse implication is similar, and is left as exercise Exercise 1.5.1. □

Recall that the *tangent line* to a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ at the point $(a, f(a))$ is given by

$$y = f'(a)(x-a) + f(a). \tag{1.3}$$

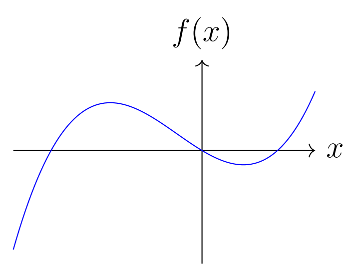
In other words, it is the graph of the affine function

$$x \mapsto f'(x_0)(x-x_0) + f(x_0). \tag{1.4}$$

Lemma 1.4.1 says that this function is the best affine approximation to the function $f(x)$ near the point a. Precisely, it says that the function (1.4) is the unique affine function such that the difference between it and $f(x)$ (the "error") goes to zero faster than $|x-a|$.

Example 1.4.2. To refresh our memory, let's find the tangent line to the function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3 + x^2 - 2x,$$



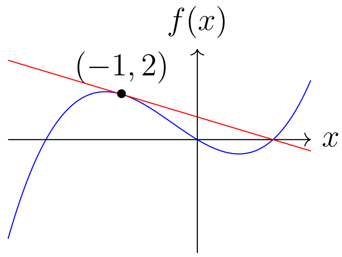
at the point $(-1, f(-1)) = (-1, 2)$. We first compute the derivative:

$$f'(x) = 3x^2 + 2x - 2, \qquad \text{hence} \qquad f'(-1) = -1.$$

The tangent line to the graph of f at the point $(-1, 2)$ is therefore given by

$$y = f'(-1)\big(x - (-1)\big) + 2 = (-1)(x+1) + 2 = 1 - x.$$

Drawing the tangent line,



we see that it is the graph of an affine function. In particular, it is the graph of the affine function

$$g(x) = -x + 1. \qquad \triangle$$

Now let's move to *two* variables. Suppose $A \subseteq \mathbb{R}^2$ is open and we have a differentiable function $f: A \to \mathbb{R}$. (We'll return to the precise definition of a differentiable function in multiple variables soon.) Remember that the *tangent plane* to the graph of f at the point $(a_1, a_2, f(a_1, a_2))$ is given by

$$\begin{aligned} z &= f_x(a_1, a_2)(x - a_1) + f_y(a_1, a_2)(y - a_2) + f(a_1, a_2) \\ &= \nabla f(a_1, a_2) \cdot (x - a_1, y - a_2) + f(a_1, a_2), \end{aligned} \tag{1.5}$$

where

$$f_x = \frac{\partial f}{\partial x} \quad \text{and} \quad f_y = \frac{\partial f}{\partial y} \tag{1.6}$$

are the *partial derivatives* of f,

$$\nabla f = (f_x, f_y) \tag{1.7}$$

is the *gradient* of f, and $\cdot$ denotes the dot product.

Example 1.4.3. Consider the function

$$f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^3 + xy - y^2 + 4.$$

Suppose we want to find the equation of the tangent plane to the graph of f (that is, the surface $z = f(x, y)$) at the point $(0, 1, f(0, 1)) = (0, 1, 3)$. We compute

$$f_x = 3x^2 + y, \qquad f_y = x - 2y.$$

Thus, the equation of the tangent plane is

$$z = f_x(0, 1)(x - 0) + f_y(0, 1)(y - 1) + f(0, 1) = x - 2(y - 1) + 3 = x - 2y + 5. \qquad \triangle$$

Exercises.

Exercises from [FRYb, §2.5]: Q5–Q7, Q10.

1.5 The derivative

We'd like to use our knowledge of linear algebra to unify and generalize the above discussion of finding a best affine approximation to a differentiable function. Let us suppose we have a function

$$f\colon \mathbb{R}^n \to \mathbb{R}^m, \qquad n, m \in \mathbb{N}.$$

It is now useful to use the notation $x_1, \dots, x_n$ for the input variables and $y_1, \dots, y_m$ for the output variables. Using vector notation, we set $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_m)$, and

$$\mathbf{y} = f(\mathbf{x}).$$

We fix a point $\mathbf{a} \in \mathbb{R}^n$ and let $\mathbf{b} = f(\mathbf{a})$. Then define

$$\Delta\mathbf{x} = \mathbf{x} - \mathbf{a}, \qquad \Delta\mathbf{y} = \mathbf{y} - \mathbf{b}.$$

Note that $\mathbf{a}$ (and thus $\mathbf{b}$) are *fixed* points, $\mathbf{x}$ is our variable, and $\mathbf{y}$ changes with $\mathbf{x}$, according to the equation $\mathbf{y} = f(\mathbf{x})$.

In our new notation, (1.3) becomes

$$\Delta\mathbf{y} = f'(\mathbf{a})\Delta\mathbf{x} \tag{1.8}$$

and (1.5) becomes

$$\Delta\mathbf{y} = f_x(\mathbf{a})(x_1 - a_1) + f_y(\mathbf{a})(x_2 - a_2) = \begin{bmatrix} f_x & f_y \end{bmatrix} \Delta\mathbf{x}, \tag{1.9}$$

where the righthand side involves matrix multiplication. These, of course, are the cases where $m = 1$ and $n = 1$ or $n = 2$, respectively. You might now be able to guess the general form.

Definition 1.5.1 (Differentiable function)**.** Let $A \subseteq \mathbb{R}^n$ be open, $f\colon A \to \mathbb{R}^m$, and $\mathbf{a} \in A$. We say f is *differentiable* at $\mathbf{a}$ if there exists a linear map $Df(\mathbf{a})\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|f(\mathbf{x}) - f(\mathbf{a}) - Df(\mathbf{a})(\mathbf{x} - \mathbf{a})\|}{\|\mathbf{x} - \mathbf{a}\|} = 0. \tag{1.10}$$

If this limit exists, then $Df(\mathbf{a})$ is called the *total derivative* of f at $\mathbf{a}$.

How do we find the total derivative in practice? As we noted in (1.1), every function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ can be written in the form

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x})), \quad \text{where} \quad f_i\colon \mathbb{R}^n \to \mathbb{R} \text{ for } i = 1, 2, \dots, m.$$

If all the partial derivatives of the components functions f_i exist, then the *Jacobian matrix*, or *matrix of partial derivatives* of f at a point $\mathbf{a} \in \mathbb{R}^n$ is the $m \times n$ matrix

$$\left[\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}.$$

Theorem 1.5.2. *If f is differentiable at $\mathbf{a}$, then its total derivative is the linear map corresponding to its Jacobian matrix (that is, the linear map given by left multiplication by the Jacobian matrix).*

We will not prove Theorem 1.5.2 in class, since the proof is a bit long and technical. As in linear algebra, we often identify a matrix and the corresponding linear map. Thus, Theorem 1.5.2 says that, if f is differentiable at $\mathbf{a}$, then

$$Df(\mathbf{a}) = \left[\tfrac{\partial f_i}{\partial x_j}(\mathbf{a})\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}.$$

As will see in Theorem 1.5.12, it is possible for the Jacobian matrix to exist (that is, the partial derivatives of all the component functions of f exist), but the function to *not* be differentiable. However, we will still use the notation $Df(\mathbf{a})$ for the Jacobian matrix in this case.

Definition 1.5.1 states that, if $f\colon A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$, is a differentiable function, then the best affine approximation to f at the point $\mathbf{a} \in A$ is given by

$$\begin{gathered} \Delta\mathbf{y} = Df(\mathbf{a})\Delta\mathbf{x}, \qquad \text{where} \\ \Delta\mathbf{y} = \mathbf{y} - \mathbf{b} = f(\mathbf{x}) - f(\mathbf{a}) \quad \text{and} \quad \Delta\mathbf{x} = \mathbf{x} - \mathbf{a}. \end{gathered} \tag{1.11}$$

Note that, when $n = m = 1$, then (1.11) becomes (1.8), describing a tangent line. On the other hand, when $n = 2$, $m = 1$, equation (1.11) becomes (1.9), describing a tangent plane. In higher dimensions, we refer to the affine approximation as a *hyperplane*.

Remark 1.5.3. Some references use the term *linear approximation* instead of *affine approximation*. However, the term *affine approximation* is more correct.

When $m = 1$, then we are considering functions

$$f\colon A \to \mathbb{R}, \quad A \subseteq \mathbb{R}^n.$$

The *gradient* of such a function is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right). \tag{1.12}$$

(When $n = 2$, this agrees with (1.7).) In this case (that is, when $m = 1$) the total derivative $Df(\mathbf{a})$ is the transpose $\nabla^\intercal f(\mathbf{a})$ of the gradient $\nabla f(\mathbf{a})$ at $\mathbf{a}$. Then (1.10) becomes

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})|}{\|\mathbf{x} - \mathbf{a}\|} = 0. \tag{1.13}$$

In fact, using the squeeze theorem (Theorem 1.2.7), we can remove the absolute value in the numerator, if desired. In other words, f is differentiable if and only if

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})}{\|\mathbf{x} - \mathbf{a}\|} = 0. \tag{1.14}$$

Example 1.5.4. Consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = 3xy^2.$$

We have

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (3y^2, 6xy), \qquad \nabla f(2,1) = (3,12).$$

Thus

$$\begin{aligned}
\lim_{(x,y)\to(2,1)} & \frac{f(x,y) - f(2,1) - \nabla f(2,1)\cdot(x-2,y-1)}{\|(x,y)-(2,1)\|} \\
&= \lim_{(x,y)\to(2,1)} \frac{3xy^2 - 6 - 3(x-2) - 12(y-1)}{\|(x-2,y-1)\|} \\
&= \lim_{(x,y)\to(2,1)} \frac{3xy^2 - 3x - 12y + 12}{\|(x-2,y-1)\|}.
\end{aligned}$$

Setting $w = x-2$, $z = y-1$, this limit is equal to

$$\lim_{(w,z)\to(0,0)} \frac{3(w+2)(z+1)^2 - 3(w+2) - 12(z+1) + 12}{\|(w,z)\|} = \lim_{(w,z)\to(0,0)} \frac{3wz^2 + 6z^2 + 6wz}{\sqrt{w^2+z^2}}.$$

Now, since $|z| = \sqrt{z^2} \le \sqrt{w^2+z^2}$, we have

$$\left|\frac{3wz^2 + 6z^2 + 6wz}{\sqrt{w^2+z^2}}\right| = \frac{|z|}{\sqrt{w^2+z^2}}|3wz + 6z + 6w| \le |3wz + 6z + 6w| \to 0$$

as $(w,z) \to (0,0)$. Thus, f is differentiable at $(2,1)$ with total derivative

$$Df(2,1) = \nabla^\intercal f(2,1) = \begin{bmatrix} 3 & 12 \end{bmatrix}.$$

and the tangent plane to the graph of f at the point $(2,1,f(2,1)) = (2,1,6)$ is given by

$$z = f(2,1) + \nabla(f)\cdot(x-2,y-1) = 6 + (3,12)\cdot(x-2,y-1) = 3x + 12y - 12. \qquad \triangle$$

Notice that, in Example 1.5.4, it was quite easy to compute the gradient ∇f, but then it required some effort to prove that the function f was differentiable. It would be nice if there were an easier way to see that a function is differentiable, at least for some large class of functions. In fact, there is, as we now explain.

First, as we saw for limits, we can actually always reduce our attention to this $m = 1$ case, as the following proposition states.

Proposition 1.5.5. *Let $A \subseteq \mathbb{R}^n$ be open, $f = (f_1, \dots, f_m)\colon A \to \mathbb{R}^m$, and $\mathbf{a} \in A$. Then f is differentiable at $\mathbf{a}$ if and only if each component function f_i, $1 \le i \le m$, is differentiable at $\mathbf{a}$. In this case, we have*

$$Df(\mathbf{a}) = \begin{bmatrix} \nabla^\intercal f_1(\mathbf{a}) \\ \vdots \\ \nabla^\intercal f_m(\mathbf{a}) \end{bmatrix}$$

where $\nabla^\intercal f_i(\mathbf{a})$ is the transpose of the gradient $\nabla f_i(\mathbf{a})$.

Proof. This follows from Definition 1.5.1 and Proposition 1.2.3. □

The following theorem (whose proof we omit) is *very* useful for showing that functions are differentiable.

Theorem 1.5.6. *Let $A \subseteq \mathbb{R}^n$ be open, $f\colon A \to \mathbb{R}$, and $\mathbf{a} \in A$. If all partial derivatives of f are continuous at $\mathbf{a}$, then f is differentiable at $\mathbf{a}$.*

Corollary 1.5.7. *Let $A \subseteq \mathbb{R}^n$ be open, $f = (f_1, \dots, f_m)\colon A \to \mathbb{R}^m$, and $\mathbf{a} \in A$. If all partial derivatives $\frac{\partial f_i}{\partial x_j}$, $1 \le i \le m$, $1 \le j \le n$, are continuous at $\mathbf{a}$, then f is differentiable at $\mathbf{a}$.*

Proof. This follows from Proposition 1.5.5 and Theorem 1.5.6. □

Example 1.5.8. Consider the function

$$f\colon \mathbb{R}^3 \to \mathbb{R}^2, \quad f(x_1, x_2, x_3) = (x_1 e^{x_2}, x_1^3 - x_2 \cos(x_3)).$$

The component functions are

$$f_1(x_1, x_2, x_3) = x_1 e^{x_2}, \quad f_2(x_1, x_2, x_3) = x_1^3 - x_2 \cos(x_3),$$

which are all differentiable functions. The total derivative is

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} e^{x_2} & x_1 e^{x_2} & 0 \\ 3x_1^2 & -\cos(x_3) & x_2 \sin(x_3) \end{bmatrix}.$$

Since all the partial derivatives are continuous, the function f is differentiable at every point of $\mathbb{R}^3$, by Theorem 1.5.6.

If $\mathbf{a} = (2, 0, \pi)$, we have

$$Df(\mathbf{a}) = \begin{bmatrix} 1 & 2 & 0 \\ 12 & 1 & 0 \end{bmatrix}.$$

Since $f(\mathbf{a}) = (2, 8)$, the best affine approximation to f at $\mathbf{a}$ is given by

$$\mathbf{y} - (2, 8) = \begin{bmatrix} 1 & 2 & 0 \\ 12 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 \\ x_3 - \pi \end{bmatrix},$$

that is,

$$\begin{aligned} y_1 &= 2 + (x_1 - 2) + 2x_2 = x_1 + 2x_2, \\ y_2 &= 8 + 12(x_1 - 2) + x_2 = 12x_1 + x_2 - 16. \end{aligned}$$

Since the graph of f lives in five dimensions, we are not able to visualize this as we can do for tangent lines and tangent planes. Nevertheless, we were still able to give an algebraic description of the approximation. △

Warning 1.5.9. The converse to Theorem 1.5.6 is false (even for single-variable functions!), as the next example shows.

Example 1.5.10. Consider the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

We have

$$\lim_{x\to 0} \frac{f(x) - 0}{x} = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$$

by the squeeze theorem (since $|x\sin(\frac{1}{x})| \le |x|$). Thus

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Therefore, f is differentiable everywhere. However, $\lim_{x\to 0} f'(x)$ does not exist, and so $f'(x)$ is not continuous at $x = 0$. △

Remark 1.5.11. A function whose partial derivatives exist and are continuous is said to be of *class* C^1. Thus, Theorem 1.5.6 and Corollary 1.5.7 say that any C^1 function is differentiable. However, Example 1.5.10 shows that there are differentiable functions that are not of class C^1.

You learned in earlier classes that, for single-variable functions, differentiability implies continuity. We have the same result in higher dimensions.

Theorem 1.5.12. *Let $A \subseteq \mathbb{R}^n$ be open, $f\colon A \to \mathbb{R}^m$, and $\mathbf{a} \in A$. If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.*

Proof. The proof is very similar to the single-variable case, and so will omit it. □

Warning 1.5.13. The *existence* of all partial derivatives is not enough to ensure that a function is differentiable, as the next example illustrates.

Example 1.5.14. Consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \begin{cases} \frac{x^2}{x-y} & x \neq y, \\ 0 & x = y. \end{cases}$$

Then

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} \frac{x^2 - 2xy}{(x-y)^2} & x \neq y, \\ 1 & x = y, \end{cases} \qquad \frac{\partial f}{\partial y}(x, y) = \begin{cases} -\frac{x^2}{(x-y)^2} & x \neq y, \\ 0 & x = y. \end{cases}$$

However, f is not continuous as $(0, 0)$. (See [FRYc, Example 2.2.9] for details.) Thus, the partial derivatives exist everywhere (and are continuous everywhere except at $(0, 0)$), but f is not differentiable at $(0, 0)$, since it is not continuous there. △

Exercises.

1.5.1. Complete the proof of Lemma 1.4.1.

1.5.2. Compute the matrix of partial derivatives of the following functions:

(a) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x,y)$

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (e^x y + \sin x, 2x, x - e^y)$

(c) $f\colon \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (2x - e^z + y, x^3 y)$

(d) $f\colon \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (xye^{xz}, y\sin x, 3xyz)$

1.5.3. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. What is the derivative of f?

1.6 Differentiation rules

The derivative in higher dimensions satisfies many of the same rules as you saw for single-variable functions.

Theorem 1.6.1. *Let $A \subseteq \mathbb{R}^n$ be open and $f, g\colon A \to \mathbb{R}^m$. Suppose that f and g are differentiable at $\mathbf{a} \in A$.*

(a) *If $c \in \mathbb{R}$, then cf is differentiable at $\mathbf{a}$ and*

$$D(cf)(\mathbf{a}) = c(Df(\mathbf{a})).$$

(b) *The function $f + g$ is differentiable at $\mathbf{a}$ and*

$$D(f+g)(\mathbf{a}) = Df(\mathbf{a}) + Dg(\mathbf{a}).$$

(c) *If $m = 1$, then fg is differentiable at $\mathbf{a}$ and*

$$D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a}).$$

This is known as the product rule.

(d) *If $m = 1$ and $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in A$, then $\frac{f}{g}$ is differentiable at $\mathbf{a}$ and*

$$D\left(\frac{f}{g}\right)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}.$$

You first encountered the *chain rule* for single-variable functions, where you learned that, for differentiable functions $f, g\colon \mathbb{R} \to \mathbb{R}$,

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

This result can be generalized to multivariable functions, using tools of linear algebra.

Theorem 1.6.2 (Chain rule)**.** *If*

$$f\colon \mathbb{R}^n \to \mathbb{R}^m \quad \textit{and} \quad g\colon \mathbb{R}^m \to \mathbb{R}^k$$

are differentiable functions and $\mathbf{a} \in \mathbb{R}^n$*, then*

$$D(g \circ f)(\mathbf{a}) = (Dg(f(\mathbf{a})))\,(Df(\mathbf{a}))\,,$$

where the product on the right-hand side is matrix multiplication (equivalently, composition of linear maps).

Example 1.6.3. Consider the functions

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad f(u,v) = (u^2+5v, 3uv), \qquad g\colon \mathbb{R}^2 \to \mathbb{R}, \quad g(x,y) = xy^2.$$

Then we have

$$Df = \begin{bmatrix} 2u & 5 \\ 3v & 3u \end{bmatrix}, \qquad Dg = \begin{bmatrix} y^2 & 2xy \end{bmatrix}.$$

Thus, using the chain rule, we have

$$\begin{aligned} D(g \circ f) &= Dg(f(u,v))Df \\ &= Dg(u^2+5v, 3uv)Df \\ &= \begin{bmatrix} (3uv)^2 & 2(u^2+5v)(3uv) \end{bmatrix} \begin{bmatrix} 2u & 5 \\ 3v & 3u \end{bmatrix} \\ &= \begin{bmatrix} 18u^3v^2 + 18(u^2+5v)uv^2 & 45u^2v^2 + 18(u^2+5v)u^2v \end{bmatrix} \\ &= \begin{bmatrix} 36u^3v^2 + 90uv^3 & 18u^4v + 135u^2v^2 \end{bmatrix}. \end{aligned}$$

In particular, we have

$$\frac{\partial(g \circ f)}{\partial u} = 36u^3v^2 + 90uv^3 \quad \text{and} \quad \frac{\partial(g \circ f)}{\partial v} = 18u^4v + 135u^2v^2.$$

Of course, we could check this by computing

$$g \circ f = (u^2+5v)(3uv)^2 = 9u^4v^2 + 45u^2v^3$$

and then calculating the partial derivatives. △

Exercises.

Exercises from [FRYb, §2.4]: Q1, Q5, Q6, Q12–Q14, Q18, Q22.

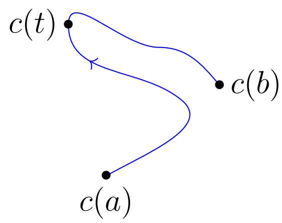
1.7 Paths, curves, and surfaces

In this section we briefly discuss paths, curves, and surfaces. We will return to these important concepts in more detail later in the course.

Definition 1.7.1 (Path, curve)**.** A *path* is a map

$$c\colon [a,b] \to \mathbb{R}^n,$$

where $a, b \in \mathbb{R}$, $a \leq b$. (We will mostly be interested in continuous paths.)

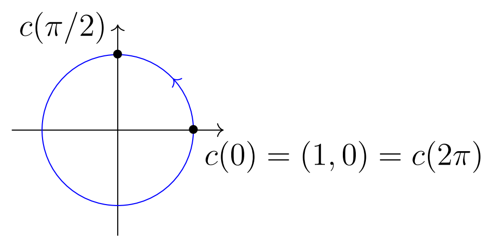


A *curve* is the image of a path:

$$C = \{c(t) : t \in [a,b]\}.$$

Example 1.7.2*.* Consider the path

$$c\colon [0, 2\pi] \to \mathbb{R}^2, \quad c(t) = (\cos(t), \sin(t)),$$



Its image is the unit circle. △

Example 1.7.3 (Parametric equation of a line)*.* If $\mathbf{p}, \mathbf{v} \in \mathbb{R}^n$, then

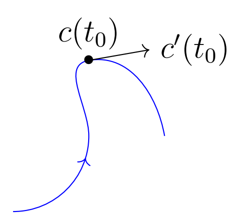
$$\ell(t) = \mathbf{p} + t\mathbf{v}, \quad t \in \mathbb{R},$$

is the *parametric equation* of the line through the point $\mathbf{p}$ and parallel to $\mathbf{v}$. If we restrict the domain to some interval $[a,b]$, then this is a curve whose image is a line segment. △

Given a path $c\colon [a,b] \to \mathbb{R}^n$, we will usually denote its derivative by $c'(t)$ instead of $Dc(t)$. Thus, if

$$c(t) = (c_1(t), \dots, c_n(t)) \quad \text{then} \quad c'(t) = (c_1'(t), \dots, c_n'(t)).$$

For $t_0 \in [a,b]$, the derivative $c'(t_0)$ is the *velocity* at time t_0. (We will often use this terminology, even if t does not really denote time in our particular situation.)

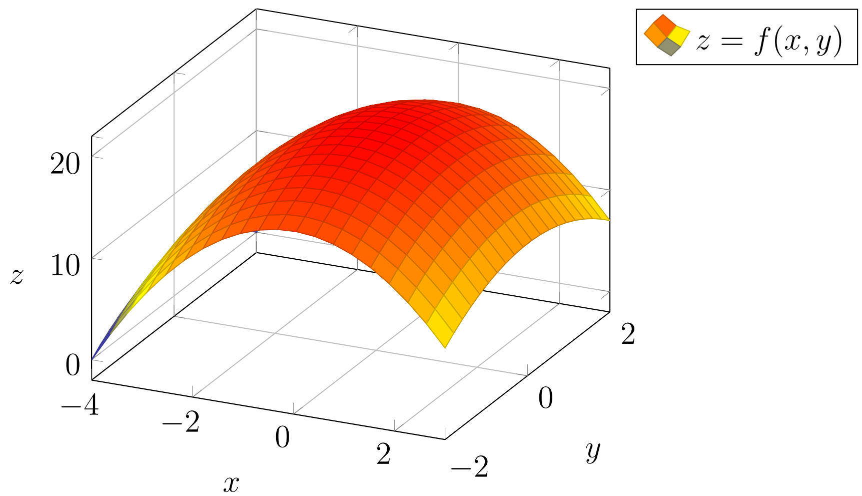


Provided $c'(t_0) \neq \mathbf{0}$, the tangent line to the curve is given by

$$\ell(t) = c(t_0) + c'(t_0)(t - t_0).$$

This is just a special case of (1.11).

Example 1.7.4. Suppose we are going on a hike. We have a differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ giving the elevation.



On a two-dimensional map (ignoring elevation), our position at time t is given by some differentiable path

$$c = (c_1, c_2)\colon [a, b] \to \mathbb{R}^2.$$

Then, the three-dimensional coordinates of our hike at time t are given by

$$d\colon [a, b] \to \mathbb{R}^3, \quad d(t) = (c_1(t), c_2(t), f(c(t))).$$

Our elevation at time t is $f(c(t))$. Using the chain rule, we have

$$(f \circ c)'(t) = Df(c(t))Dc(t) = \begin{bmatrix} \frac{\partial f}{\partial x}(c(t)) & \frac{\partial f}{\partial y}(c(t)) \end{bmatrix} \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \nabla f(c(t)) \cdot c'(t).$$

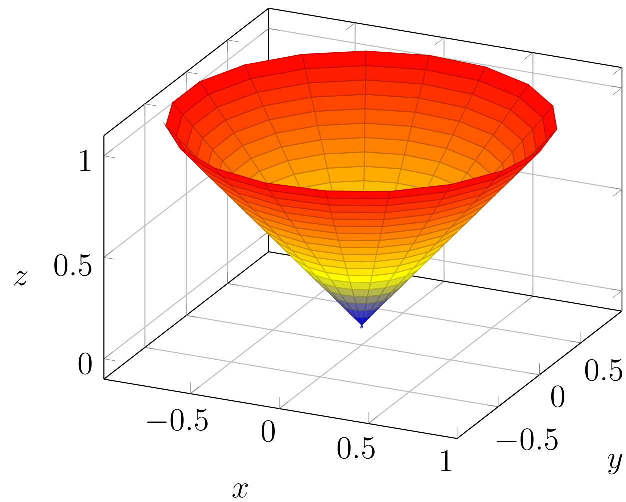
This gives us the rate of change of elevation at time t. △

A surface is a two-dimensional object in three-dimensional space. There are three ways of describing a surface:

(a) We can give a *parameterization* of the surface. For example,

$$\Phi\colon [0, 1] \times [0, 2\pi] \to \mathbb{R}^3, \quad \Phi(u, v) = (u\cos(v), u\sin(v), u)$$

describes a cone:



We will discuss parameterizations of surfaces in more detail in Section 7.1.

(b) We can describe the surface as the set of solutions to an equation. For example,

$$S = \{(x, y, z) : x^2 + y^2 \leq 1,\ z^2 = x^2 + y^2\}$$

is the same cone as above.

(c) We can describe a surface as the graph of a function $f\colon A \to \mathbb{R}$. For example, the graph of the function

$$f\colon A \to \mathbb{R}, \quad f(x, y) = \sqrt{x^2 + y^2}, \quad A = \{(x, y) : x^2 + y^2 \leq 1\}$$

is the same cone as above.

If $A \subseteq \mathbb{R}^n$, $f\colon A \to \mathbb{R}$, and $k \in \mathbb{R}$, then the *level set* of f at k is the set

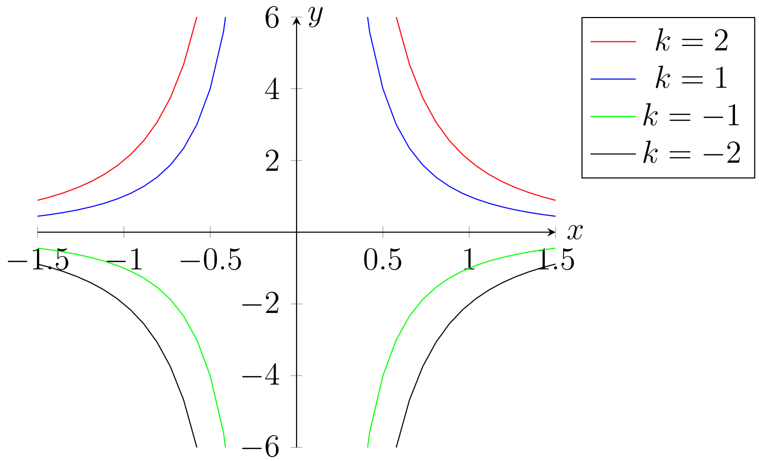
$$\{\mathbf{x} \in A : f(\mathbf{x}) = k\}. \tag{1.15}$$

When $n = 2$, level sets are sometimes called *contour lines*. A *contour plot* is an image showing multiple contour lines for the same function.

Example 1.7.5. Consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^2 y.$$

The level set of f at k is the set of all points (x, y) such that $k = f(x, y) = x^2 y$, or, equivalently, $y = \frac{k}{x^2}$.



△

Exercises.

Exercises from [FRYb, §1.6]: Q1–Q4, Q6, Q10, Q21.

Exercises from [FRYb, §1.7]: Q2–Q11.

1.8 Directional derivatives

A good reference for the material in this section is [FRYc, §2.7]. Throughout this section, $A \subseteq \mathbb{R}^n$ is an open set and $f\colon A \to \mathbb{R}$.

Definition 1.8.1 (Directional derivative)**.** Let $\mathbf{a} \in A$, and $\mathbf{v} \in \mathbb{R}^n$. The *directional derivative of f at $\mathbf{a}$ along $\mathbf{v}$* is

$$D_{\mathbf{v}} f(\mathbf{a}) := \left.\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{a} + t\mathbf{v})\right|_{t=0}.$$

Remark 1.8.2. Some references require the vector $\mathbf{v}$ in Definition 1.8.1 to be a unit vector, that is, $\|\mathbf{v}\| = 1$. We will not require this.

Example 1.8.3. Consider the function

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) = xy^2 - 3z,$$

and let $\mathbf{a} = (q,r,s)$, $\mathbf{v} = (2,1,-1)$. Then we have

$$\begin{aligned} f(\mathbf{a} + t\mathbf{v}) &= f(q+2t, r+t, s-t) \\ &= (q+2t)(r+t)^2 - 3(s-t) \\ &= qr^2 + 2tr^2 + 2qtr + 4t^2r + qt^2 + 2t^3 - 3s + 3t. \end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{a} + t\mathbf{v}) = 2r^2 + 2qr + 8tr + 2qt + 6t^2 + 3,$$

and so

$$D_{\mathbf{v}} f(q, r, s) = \left. \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{a} + t\mathbf{v}) \right|_{t=0} = 2r^2 + 2qr + 3. \qquad \triangle$$

Proposition 1.8.4. *If f is differentiable at $\mathbf{a}$, then*

$$D_{\mathbf{v}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}.$$

Proof. Let

$$\ell(t) = \mathbf{a} + t\mathbf{v}, \quad t \in \mathbb{R}.$$

Then

$$\begin{aligned} D_{\mathbf{v}} f(\mathbf{a}) &= \left. \frac{\mathrm{d}}{\mathrm{d}t} f \circ \ell(t) \right|_{t=0} \\ &= (f \circ \ell)'(t) \\ &= Df(\ell(0)) D\ell(0) && \text{(chain rule)} \\ &= \nabla^{\mathsf{T}} f(\mathbf{a}) \ell'(0) \\ &= \nabla f(\mathbf{a}) \cdot \mathbf{v}. && \square \end{aligned}$$

Example 1.8.5. Returning to the function $f(x, y, z) = xy^2 - 3z$ of Example 1.8.3, we have

$$\nabla f(x, y, z) = (y^2, 2xy, -3).$$

Thus

$$D_{(2,1,-1)} f(x, y, z) = 2y^2 + 2xy + 3.$$

Taking $(x, y, z) = (q, r, s)$, this gives the same directional derivative that we computed in Example 1.8.3. $\triangle$

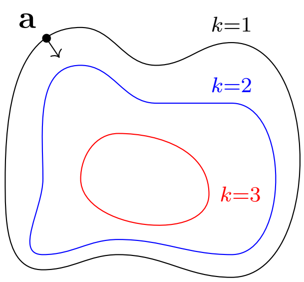
Note that there is a significant computational advantage that comes from Proposition 1.8.4. Namely, we can compute the gradient ∇f *once*, and then find *any* directional derivative by computing a simple dot product.

The following result gives two important interpretations of the gradient.

Proposition 1.8.6. *Suppose f is differentiable at $\mathbf{a} \in A$, and let $k = f(\mathbf{a})$.*

(a) *The gradient $\nabla f(\mathbf{a})$ is a vector pointing in the direction in which f increases most rapidly.*

(b) *If S is the level set of f at k (hence $\mathbf{a} \in S$), then $\nabla f(\mathbf{a})$ is orthogonal to the tangent line/plane/hyperplane to S at $\mathbf{a}$.*



Proof. (a) For any unit vector $\mathbf{v}$, $D_{\mathbf{v}}f(\mathbf{a})$ tells us how fast f changes along the direction $\mathbf{v}$. We have

$$D_{\mathbf{v}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v} = \|\mathbf{v}\| \, \|\nabla f(\mathbf{a})\| \cos\theta = \|\nabla f(\mathbf{a})\| \cos\theta,$$

where θ is the angle between $\nabla f(\mathbf{a})$ and $\mathbf{v}$. Now, $\cos\theta$ attains its maximal value when $\theta = 0$, that is, when $\nabla f(\mathbf{a})$ and $\mathbf{v}$ are parallel.

(b) Let $c(t)$ be a path in S such that $c(t_0) = \mathbf{a}$. Then $c'(t_0)$ is a tangent line contained in the tangent plane to S at $\mathbf{a}$. So we want to show that $\nabla f(\mathbf{a}) \perp c'(t_0)$ for *any* such path.

For all t we have

$$c(t) \in S \implies f(c(t)) = k \implies (f \circ c)'(t) = 0.$$

Thus we have

$$\begin{aligned} 0 &= (f \circ c)'(t) \\ &= D(f \circ c)(t) \\ &= \nabla f(c(t)) \cdot c'(t). && \text{(chain rule)} \end{aligned}$$

Thus, taking $t = t_0$, we have $0 = \nabla f(\mathbf{a}) \cdot c'(t_0)$, and so $\nabla f(\mathbf{a}) \perp c'(t_0)$. $\square$

Exercises.

Exercises from [FRYb, §2.7]: Q1–Q8, Q10–Q20.

1.9 Higher order derivatives

A good reference for the material in this section is [FRYc, §2.3].

Suppose $A \subseteq \mathbb{R}^n$ is open and $f\colon A \to \mathbb{R}$. If the partial derivatives $\frac{\partial f}{\partial x_i}$ exist, then they are again functions from $A \to \mathbb{R}$, and we can try to differentiate again. If the partial derivatives are differentiable, we obtain the *second order derivatives*

$$\frac{\partial^2 f}{\partial x_i \, \partial x_j} := \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j}, \qquad 1 \le i, j \le n.$$

Of course, if the partial derivatives of these second order derivatives exist, we can differentiate again, to get the *third order derivatives*

$$\frac{\partial^3 f}{\partial x_i \, \partial x_j \, \partial x_k} := \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x_j \, \partial x_k}, \qquad 1 \le i, j, k \le n.$$

We can continue as long as we obtain functions whose partial derivatives exist.

We may also use the subscript notation of (1.6), especially when our variables are denoted x, y, z. Then, for example, we have

$$f_{xx} := (f_x)_x = \frac{\partial^2 f}{\partial x \, \partial x}, \qquad f_{xy} := (f_x)_y = \frac{\partial^2 f}{\partial y \, \partial x},$$
$$f_{yx} := (f_y)_x = \frac{\partial^2 f}{\partial x \, \partial y}, \qquad f_{yy} := (f_y)_y = \frac{\partial^2 f}{\partial y \, \partial y}.$$

Pay careful attention to the order of the subscripts!

Example 1.9.1. Suppose $\alpha, \beta \in \mathbb{R}$, and let $f(x, y) = e^{\alpha x + \beta y}$. Then

$$f_x = \frac{\partial f}{\partial x} = \alpha e^{\alpha x + \beta y}, \qquad f_y = \frac{\partial f}{\partial y} = \beta e^{\alpha x + \beta y},$$
$$f_{xx} = \frac{\partial^2 f}{\partial x \, \partial x} = \alpha^2 e^{\alpha x + \beta y}, \qquad f_{xy} = \frac{\partial^2 f}{\partial y \, \partial x} = \alpha\beta e^{\alpha x + \beta y},$$
$$f_{yx} = \frac{\partial^2 f}{\partial x \, \partial y} = \alpha\beta e^{\alpha x + \beta y}, \qquad f_{yy} = \frac{\partial^2 f}{\partial y \, \partial y} = \beta^2 e^{\alpha x + \beta y}.$$

In general, for $m, n \ge 0$, we have

$$\frac{\partial^{m+n} f}{\partial x^m \, \partial y^n} = \alpha^m \beta^n e^{\alpha x + \beta y}. \tag{1.16}$$

Try proving this by induction (Exercise 1.9.1). △

Example 1.9.2. Suppose

$$f(x_1, x_2, x_3, x_4) = x_1^3 x_2 \cos(x_3 x_4).$$

Then

$$\begin{aligned}
\frac{\partial^4 f}{\partial x_3 \, \partial x_1 \, \partial x_2 \, \partial x_4} &= \frac{\partial^3}{\partial x_3 \, \partial x_1 \, \partial x_2} \big(-x_1^3 x_2 x_3 \sin(x_3 x_4)\big) \\
&= \frac{\partial^2}{\partial x_3 \, \partial x_1} \big(-x_1^3 x_3 \sin(x_3 x_4)\big) \\
&= \frac{\partial}{\partial x_3} \big(-3x_1^2 x_3 \sin(x_3 x_4)\big) \\
&= -3x_1^2 \sin(x_3 x_4) - 3x_1^2 x_3 x_4 \cos(x_3 x_4).
\end{aligned}$$

We also have

$$\frac{\partial^4 f}{\partial x_1 \, \partial x_3 \, \partial x_4 \, \partial x_2} = \frac{\partial^3 f}{\partial x_1 \, \partial x_3 \, \partial x_4} \big(x_1^3 \cos(x_3 x_4)\big)$$

$$
\begin{aligned}
&= \frac{\partial^2 f}{\partial x_1\,\partial x_3}\left(-x_1^3 x_3 \sin(x_3x_4)\right) \\
&= \frac{\partial f}{\partial x_1}\left(-x_1^3 \sin(x_3x_4) - x_1^3 x_3 x_4 \cos(x_3x_4)\right) \\
&= -3x_1^2 \sin(x_3x_4) - 3x_1^2 x_3 x_4 \cos(x_3x_4).
\end{aligned}
$$
△

Definition 1.9.3 (Class C^k, class C^∞)**.** For $k \in \mathbb{N}$, a function $f\colon A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$, is *of class* C^k *at* $\mathbf{a}$ if all of the k-th order partial derivatives exist and are continuous at $\mathbf{a}$. More generally, $f = (f_1, \dots, f_m)\colon A \to \mathbb{R}^m$ is of class C^k at $\mathbf{a}$ if each component function f_i, $1 \le i \le m$, is of class C^k at $\mathbf{a}$. A function is *of class* C^∞ *at* $\mathbf{a}$ if it is of class C^k at $\mathbf{a}$ for all $k \in \mathbb{N}$. We say a function is *of class* C^k (respectively, *of class* C^∞) it is of class C^k (respectively, of class C^∞) at all $\mathbf{a} \in A$.

Example 1.9.4. (a) A function is of class C^0 if it is continuous.

(b) A function is C^1 if all partial derivatives exist and are continuous. By Theorem 1.5.6, all C^1 functions are differentiable.

We can think of the condition of being a class C^k function as a "higher" continuity or differentiability condition. △

Example 1.9.5. The functions $\sin x$, $\cos x$, e^x, $\frac{1}{x}$, and $\log x$ are all of class C^∞. (Note that we are implicitly assuming that the domain of each function is the subset of $\mathbb{R}$ where it is defined.) Similarly, all polynomials are of class C^∞. △

We saw in Example 1.9.1 that $f_{xy} = f_{yx}$, and we saw in Example 1.9.2 that

$$\frac{\partial^4 f}{\partial x_3\,\partial x_1\,\partial x_2\,\partial x_4} = \frac{\partial^4 f}{\partial x_1\,\partial x_3\,\partial x_4\,\partial x_2}.$$

In both examples, it did not matter in what order we computed the partial derivatives. The following theorem explains this phenomenon.

Theorem 1.9.6 (Clairaut's theorem or Schwarz's theorem)**.** *Suppose $A \subseteq \mathbb{R}^n$ is open and $f\colon A \to \mathbb{R}$. If the partial derivatives $\frac{\partial^2 f}{\partial x_i\,\partial x_j}$ and $\frac{\partial^2 f}{\partial x_j\,\partial x_i}$ exist and are continuous at $\mathbf{a}$, then*

$$\frac{\partial^2 f}{\partial x_i\,\partial x_j}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_j\,\partial x_i}(\mathbf{a}).$$

In particular, if f is of type C^2, then

$$\frac{\partial^2 f}{\partial x_i\,\partial x_j} = \frac{\partial^2 f}{\partial x_j\,\partial x_i} \quad \textit{for all } 1 \le i, j \le n.$$

Proof. A proof of this theorem can be found in [FRYc, §2.3.1]. □

Remark 1.9.7. Using Theorem 1.9.6 and induction, we get analogues for higher order derivatives. For example, if $f\colon \mathbb{R}^3 \to \mathbb{R}$ is of class C^4, then

$$\frac{\partial^4 f}{\partial x\,\partial y\,\partial x\,\partial z} = \frac{\partial^4 f}{\partial z\,\partial y\,\partial x^2}.$$

Warning 1.9.8. There exist functions whose mixed partial derivatives exist, but are not equal!

Example 1.9.9. Consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then one can show that

$$\frac{\partial^2 f}{\partial x\, \partial y}(0,0) = 1 \quad \text{and} \quad \frac{\partial^2 f}{\partial y\, \partial x}(0,0) = -1.$$

See [FRYc, §2.3.2] for the details of this computation. △

The class of C^k functions is closed under the following operations:

- *Addition.* If $f, g\colon A \to \mathbb{R}$ are of class C^k, then so is $f + g$.
- *Scalar multiplication.* If $f\colon A \to \mathbb{R}$ is of class C^k and $\alpha \in \mathbb{R}$, then αf is of class C^k.
- *Multiplication.* If $f, g\colon A \to \mathbb{R}$ are of class C^k, then so is fg.
- *Composition.* If $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$,

  $$f\colon A \to B, \qquad g\colon B \to \mathbb{R}^m,$$

  and f, g are both of class C^k, then so is $g \circ f$.

Exercises.

1.9.1. Prove (1.16) using induction.

Exercises from [FRYb, §2.3]: Q1–Q7.

1.10 Taylor's theorem

Recall the single-variable version of *Taylor's theorem*. It states that, if $f\colon \mathbb{R} \to \mathbb{R}$ is of class C^k at a, then

$$f(a+h) = \underbrace{f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \cdots + \frac{1}{k!}f^{(k)}(a)h^k}_{k\text{-th Taylor polynomial}} + R_k(h),$$

where

$$\frac{R_k(h)}{h^k} \to 0 \text{ as } h \to 0.$$

For a refresher on this topic, see [FRYa, §3.4].

When $k = 1$, Taylor's theorem corresponds to the one-variable affine approximations we discussed in Section 1.4. When $k = 2$, it yields the best possible *quadratic* approximation to the function.

We would like to generalize Taylor's theorem to more variables. For the remainder of this section, suppose $A \subseteq \mathbb{R}^n$ is open and $f\colon A \to \mathbb{R}$.

Definition 1.10.1 (Taylor polynomial)**.** If the k-th order partial derivatives of f exist at $\mathbf{a}$, then the *k-th Taylor polynomial*, or the *degree k Taylor polynomial* of f at $\mathbf{a}$ is

$$p_k(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \sum_{i=1}^{k} \frac{1}{i!} \sum_{j_1,\dots,j_i=1}^{n} \frac{\partial^i f}{\partial x_{j_1} \cdots \partial x_{j_i}}(\mathbf{a}) h_{j_1} \cdots h_{j_i}, \tag{1.17}$$

where $\mathbf{h} = (h_1, \dots, h_n)$. We define the *remainder function*

$$R_{\mathbf{a},k}(\mathbf{h}) := f(\mathbf{a}+\mathbf{h}) - p_k(\mathbf{a}+\mathbf{h}),$$

so that

$$f(\mathbf{a}+\mathbf{h}) = p_k(\mathbf{a}+\mathbf{h}) + R_{\mathbf{a},k}(\mathbf{h}).$$

Theorem 1.10.2 (Taylor's theorem)**.** *If f is of class C^k at $\mathbf{a}$, then*

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{R_{\mathbf{a},k}(\mathbf{h})}{\|\mathbf{h}\|^k} = 0.$$

Proof. We will not prove this theorem in class. A proof of the $k = 2$ case can be found in [MT11, §3.2, Th. 3]. □

Let us now describe the $k = 2$ case of Taylor's theorem more explicitly.

Definition 1.10.3 (Hessian matrix)**.** If the second partial derivatives of f exist, then the *Hessian matrix* of f at $\mathbf{a}$ is

$$\mathbf{H}f(\mathbf{a}) := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_1\,\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n}(\mathbf{a}) \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_n\,\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{a}) \end{bmatrix} \in \mathrm{Mat}_{n\times n}(\mathbb{R}). \tag{1.18}$$

Comparing (1.17) and (1.18), we see that the degree 2 Taylor polynomial is given by

$$p_2(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h}^\intercal(\mathbf{H}f(\mathbf{a}))\mathbf{h}, \tag{1.19}$$

where $\mathbf{h}^\intercal = \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix}$ is the transpose of $\mathbf{h}$.

Corollary 1.10.4 (Second-order Taylor formula)**.** *If f is of class C^2 at* $\mathbf{a}$*, then*

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h}^\intercal(\mathbf{H}f)\mathbf{h} + R_{\mathbf{a},2}(\mathbf{h}),$$

where

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{R_{\mathbf{a},2}(\mathbf{h})}{\|\mathbf{h}\|^2} = 0.$$

Example 1.10.5. Consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = e^y \cos(x^2 - 2y).$$

Let us find the degree 2 Taylor polynomial at $\mathbf{a} = (\sqrt{\pi}, \pi)$. We have

$$f(\sqrt{\pi}, \pi) = e^\pi \cos(-\pi) = -e^\pi.$$

Next we compute the first-order partial derivatives:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -2e^y x \sin(x^2 - 2y), & \frac{\partial f}{\partial x}(\sqrt{\pi}, \pi) &= -2e^\pi \sqrt{\pi} \sin(-\pi) = 0,\\
\frac{\partial f}{\partial y} &= e^y \cos(x^2 - 2y) + 2e^y \sin(x^2 - 2y), & \frac{\partial f}{\partial y}(\sqrt{\pi}, \pi) &= e^\pi \cos(-\pi) + 2e^\pi \sin(-\pi) = -e^\pi.
\end{aligned}$$

Finally, we compute the second-order partial derivatives:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -2e^y \left(\sin(x^2 - 2y) + 2x^2 \cos(x^2 - 2y)\right), & \frac{\partial^2 f}{\partial x^2}(\sqrt{\pi}, \pi) &= 4\pi e^\pi,\\
\frac{\partial^2 f}{\partial y\, \partial x} &= -2x \left(e^y \sin(x^2 - 2y) - 2e^y \cos(x^2 - 2y)\right), & \frac{\partial^2 f}{\partial y\, \partial x}(\sqrt{\pi}, \pi) &= -4\sqrt{\pi} e^\pi,\\
\frac{\partial^2 f}{\partial y^2} &= -3e^y \cos(x^2 - 2y) + 4e^y \sin(x^2 - 2y), & \frac{\partial^2 f}{\partial y^2}(\sqrt{\pi}, \pi) &= 3e^\pi,\\
\frac{\partial^2 f}{\partial x\, \partial y} &= -2x \left(e^y \sin(x^2 - 2y) - 2e^y \cos(x^2 - 2y)\right), & \frac{\partial^2 f}{\partial x\, \partial y}(\sqrt{\pi}, \pi) &= -4\sqrt{\pi} e^\pi.
\end{aligned}$$

Thus, the gradient and Hessian are

$$\nabla f(\sqrt{\pi}, \pi) = (0, -e^\pi), \qquad \mathbf{H}f(\sqrt{\pi}, \pi) = \begin{bmatrix} 4\pi e^\pi & -4\sqrt{\pi} e^\pi \\ -4\sqrt{\pi} e^\pi & 3e^\pi \end{bmatrix},$$

and the degree 2 Taylor polynomial is

$$\begin{aligned}
&p_2\left((\sqrt{\pi}, \pi) + (h_1, h_2)\right)\\
&\quad = -e^\pi + (0, -e^\pi) \cdot (h_1, h_2) + \frac{1}{2} \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} 4\pi e^\pi & -4\sqrt{\pi} e^\pi \\ -4\sqrt{\pi} e^\pi & 3e^\pi \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\\
&\quad = -e^\pi - e^\pi h_2 + 2\pi e^\pi h_1^2 + \frac{3}{2} e^\pi h_2^2 - 4\sqrt{\pi} e^\pi h_1 h_2.
\end{aligned}$$

Since f is of class C^2, Corollary 1.10.4 tells us that the difference between the function and this polynomial goes to zero faster than $\|\mathbf{h}\|^2$ as $\mathbf{h} \to \mathbf{0}$. △

Exercises.

1.10.1 ([MT11, §3.2, Exercise 1]). Let $f(x, y) = e^{x+y}$.

(a) Find the first-order Taylor polynomial for f at $(0, 0)$.

(b) Find the second-order Taylor polynomial for f at $(0, 0)$.

1.10.2 ([MT11, §3.2, Exercise 2]). Suppose

$$L\colon \mathbb{R}^2 \to \mathbb{R}, \quad L(x, y) = ax + by,$$

is a linear function.

(a) Find the first-order Taylor approximation for L.
Thus $p_1 = L$.

(b) Find the second-order Taylor approximation for L.

(c) What will the higher-order approximations look like?

1.10.3 ([MT11, §3.2, Exercises 3–7]). For each of the following, determine the second-order Taylor polynomial for the given function about the point $(0, 0)$. Use this polynomial to approximate $f(-1, -1)$. Compare your approximation to the exact value using a calculator.

(a) $f(x, y) = (x + y)^2$

(b) $f(x, y) = 1/(x^2 + y^2 + 1)$

(c) $f(x, y) = e^{x+y}$

(d) $f(x, y) = e^{-x^2-y^2} \cos(xy)$

1.10.4 ([MT11, §3.2, Exercise 11]). Let $g(x, y) = \sin(xy) - 3x^2 \ln y + 1$. Find the degree 2 polynomial that best approximates g near the point $(\pi/2, 1)$.

1.11 The implicit function theorem

In this section we state the *implicit function theorem*, which is one of the most important theorems in mathematical analysis. A detailed proof is beyond the scope of this course, but we will give some intuition for why it is true. A good reference for the material in this section is [MT11, §3.5].

Recall that if $A \subseteq \mathbb{R}^n$, $f\colon \mathbb{R}^n \to \mathbb{R}$, and $k \in \mathbb{R}$, then the level set of f at k is

$$\{\mathbf{x} \in A : f(\mathbf{x}) = k\}$$

If we define $g\colon A \to \mathbb{R}$ by $g(\mathbf{x}) = f(\mathbf{x}) - k$, then

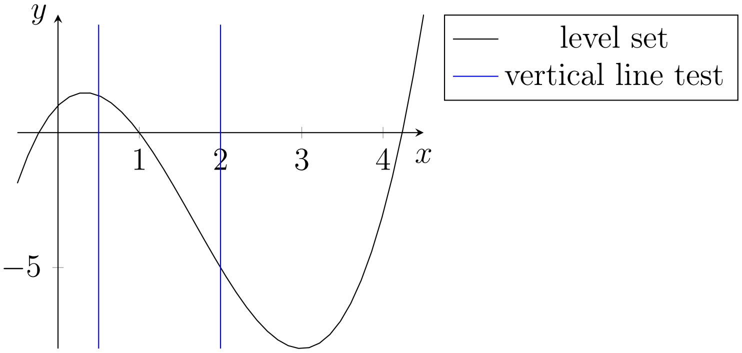
$$\{\mathbf{x} \in A : f(\mathbf{x}) = k\} = \{\mathbf{x} \in A : g(\mathbf{x}) = 0\}.$$

Thus, without loss of generality, we can assume that $k = 0$.

Now, consider such a level set, given by the equation

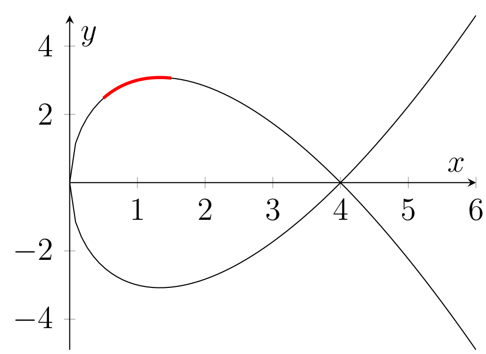
$$f(x, y) = 0.$$

When can we solve for one variable in terms of another? If the level set satisfies the vertical line test (every vertical line meets the graph at most once), then we can write y as a function of x.



Now consider the level set given by the equation

$$f(x, y) = x^3 - 8x^2 + 16x - y^2 = 0.$$



This level set does not satisfy the vertical line test. However, near some points we could write y as a function of x. For instance, consider the portion of the level set near the point $(1, 3)$, indicated in red above. In this small region, we *can* write y as a function of x. On the other hand, we would *not* be able to do this near the point $(0, 0)$. The problem is that the tangent line is vertical there.

Definition 1.11.1 (Neighbourhood)**.** A *neighbourhood* of a point $\mathbf{a} \in \mathbb{R}^n$ is an open set $U \subseteq \mathbb{R}^n$ such that $\mathbf{a} \in U$.

To state the implicit function theorem, we use the fact that $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ to write points of $\mathbb{R}^{n+1}$ in the form $(\mathbf{x}, z)$, where $\mathbf{x} \in \mathbb{R}^n$, and $z \in \mathbb{R}$. In particular, if $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$, then $\frac{\partial F}{\partial z}$ denotes the partial derivative with respect to the last variable.

Theorem 1.11.2 (Special implicit function theorem)**.** *Suppose $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is of class C^1 and $(\mathbf{x}_0, z_0) \in \mathbb{R}^{n+1}$, such that*

$$F(\mathbf{x}_0, z_0) = 0 \quad \textit{and} \quad \frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0.$$

Then there exist neighbourhoods $U \subseteq \mathbb{R}^n$ of $\mathbf{x}_0$ and $V \subseteq \mathbb{R}$ of z_0 such that there is a unique function

$$g\colon U \to V \quad \textit{such that} \quad F(\mathbf{x}, g(\mathbf{x})) = 0 \textit{ for all } \mathbf{x} \in U.$$

Furthermore, g is of class C^1 and

$$\frac{\partial g}{\partial x_i}(\mathbf{x}) = -\frac{\frac{\partial F}{\partial x_i}(\mathbf{x}, g(\mathbf{x}))}{\frac{\partial F}{\partial z}(\mathbf{x}, g(\mathbf{x}))} \tag{1.20}$$

for all $\mathbf{x} \in U$ and $i \in \{1, \dots, n\}$.

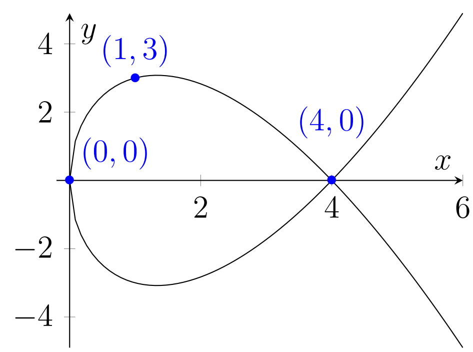
Proof. This theorem is a special case of the general implicit function we will state below (Theorem 1.11.5). Once it is known that $z = g(\mathbf{x})$ exists and is differentiable, we can obtain equation (1.20) by implicit differentiation: Applying the chain rule to $F(\mathbf{x}, g(\mathbf{x})) = 0$ gives

$$\frac{\partial F}{\partial x_i}(\mathbf{x}, g(\mathbf{x})) + \left(\frac{\partial F}{\partial z}(\mathbf{x}, g(\mathbf{x}))\right) \frac{\partial g}{\partial x_i}(\mathbf{x}) = 0,$$

which is equivalent to (1.20). □

Example 1.11.3. Consider the points $(1, 3)$, $(0, 0)$, and $(4, 0)$ of the level set

$$F(x, y) = x^3 - 8x^2 + 16x - y^2 = 0.$$



We have

$$\frac{\partial F}{\partial y} = -2y.$$

This is zero if and only if $y = 0$. Therefore, the special implicit function theorem (Theorem 1.11.2) implies that we can write y as a function of x near every point on the level set where $y \neq 0$. In particular, we can do so near the point $(1, 3)$. (Note that the z in Theorem 1.11.2 is our y here.)

We could also interchange the roles of the variables, and try to write x as a function of y. Then the z of Theorem 1.11.2 is our x. Since

$$\frac{\partial F}{\partial x} = 3x^2 - 16x + 16 = (x - 4)(3x - 4),$$

we can write x as a function of y near every point on the level set where $x \neq \frac{4}{3}, 4$. In particular, we can do this near $(0, 0)$, a point where we could *not* write y as a function of x.

The most problematic point is $(4, 0)$, since both partial derivatives of F vanish here:

$$\nabla F(4, 0) = \mathbf{0}.$$

Here we cannot write any variable as a function of the others. Looking at the picture of the level set above, we see why this is the case. The level set does not satisfy the vertical line test or the horizontal line test in *any* neighbourhood of the point $(4,0)$. △

Remark 1.11.4. As illustrated in Example 1.11.3, if $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is of class C^1 and $\mathbf{x}_0 \in \mathbb{R}^{n+1}$, such that

$$F(\mathbf{x}_0) = 0 \quad \text{and} \quad \nabla F(\mathbf{x}_0) \neq \mathbf{0},$$

then we can write at least one variable in terms of the others.

We now give a more general version of the implicit function theorem. Suppose we have a C^1 function

$$F = (F_1, \dots, F_m)\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m.$$

We write points of $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ as $(\mathbf{x}, \mathbf{z})$, with $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{R}^m$. Define

$$D_{\mathbf{x}}F := \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix} \in \mathrm{Mat}_{m\times n}(\mathbb{R}) \qquad \text{and}$$

$$D_{\mathbf{z}}F := \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial z_1} & \cdots & \frac{\partial F_m}{\partial z_m} \end{bmatrix} \in \mathrm{Mat}_{m\times m}(\mathbb{R}).$$

(This conflicts with our notation for the directional derivative, but the context should make clear what we mean.) We view the equation $F(\mathbf{x}, \mathbf{z}) = \mathbf{0}$ as a collection of m equations:

$$F_1(\mathbf{x}, \mathbf{z}) = 0, \ \dots, \ F_m(\mathbf{x}, \mathbf{z}) = 0.$$

We're interested in trying to solve for $z_1, \dots, z_m$ in terms of $x_1, \dots, x_n$.

Theorem 1.11.5 (General implicit function theorem)**.** *Suppose $F\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 function and $(\mathbf{x}_0, \mathbf{z}_0) \in \mathbb{R}^n \times \mathbb{R}^m$, such that*

$$F(\mathbf{x}_0, \mathbf{z}_0) = \mathbf{0} \quad \textit{and} \quad D_{\mathbf{z}}F(\mathbf{x}_0, \mathbf{z}_0) \textit{ is invertible.}$$

Then there exist neighbourhoods $U \subseteq \mathbb{R}^n$ of $\mathbf{x}_0$ and $V \subseteq \mathbb{R}^m$ of $\mathbf{z}_0$ such that there is a unique function

$$g\colon U \to V \quad \textit{such that} \quad F(\mathbf{x}, g(\mathbf{x})) = \mathbf{0} \textit{ for all } \mathbf{x} \in U.$$

Furthermore, g is of class C^1 and

$$Dg(\mathbf{x}) = -\big(D_{\mathbf{z}}F(\mathbf{x}, g(\mathbf{x}))\big)^{-1} D_{\mathbf{x}}F(\mathbf{x}, g(\mathbf{x})).$$

Sketch of proof. For a complete proof of this theorem, see [Gou59, p. 45]. We give here only a sketch of a proof to give some intuition as to why the theorem might hold. The first degree Taylor polynomial of F at $(\mathbf{x}, \mathbf{z})$ (see (1.17)) gives the approximation

$$F(\mathbf{x}, \mathbf{z}) \approx F(\mathbf{x}_0, \mathbf{z}_0) + \nabla F(\mathbf{x}, \mathbf{z}) \cdot \big((\mathbf{x}, \mathbf{z}) - (\mathbf{x}_0, \mathbf{z}_0)\big)$$

$$= F(\mathbf{x}_0, \mathbf{z}_0) + D_{\mathbf{x}}F(\mathbf{x}_0, \mathbf{z}_0)(\mathbf{x} - \mathbf{x}_0) + D_{\mathbf{z}}F(\mathbf{x}_0, \mathbf{z}_0)(\mathbf{z} - \mathbf{z}_0).$$

If $D_{\mathbf{z}}F(\mathbf{x}_0, \mathbf{z}_0)$ is invertible, then we can (approximately) solve the equation $F(\mathbf{x}, \mathbf{z}) = \mathbf{0}$ to get

$$\mathbf{z} \approx \mathbf{z}_0 - (D_{\mathbf{z}}F(\mathbf{x}_0, \mathbf{z}_0))^{-1} \left(F(\mathbf{x}_0, \mathbf{z}_0) + D_{\mathbf{x}}F(\mathbf{x}_0, \mathbf{z}_0)(\mathbf{x} - \mathbf{x}_0) \right).$$

Even though this is only an approximation, it is still true that if $D_{\mathbf{z}}F(\mathbf{x}_0, \mathbf{z}_0)$ is invertible, then we can solve for $\mathbf{z}$ in terms of $\mathbf{x}$ at nearby points. □

Remark 1.11.6. Remember that a square matrix is invertible if and only if its determinant is nonzero. So we can see if $D_{\mathbf{z}}F(\mathbf{x}_0, \mathbf{z}_0)$ is invertible or not by computing its determinant.

Example 1.11.7. Consider the set of points (x, y, z) satisfying the following two equations:

$$x \sin y + e^{x+z} = 1,$$
$$x^2 y + e^{yz} - z^2 = 1.$$

Note that the points

$$(1, 0, -1) \qquad \text{and} \qquad (-1, 0, 1)$$

are both solutions. Can we solve for y and z in terms of x around these points?

We will use the implicit function theorem (Theorem 1.11.5) with $n = 1$, $m = 2$, and $F = (F_1, F_2) \colon \mathbb{R}^3 \to \mathbb{R}^2$, with

$$F_1(x, y, z) = x \sin y + e^{x+z} - 1, \qquad F_2(x, y, z) = x^2 y + e^{yz} - z^2 - 1.$$

We have

$$DF = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} = \begin{bmatrix} \sin y + e^{x+z} & x \cos y & e^{x+z} \\ 2xy & x^2 + z e^{yz} & y e^{yz} - 2z \end{bmatrix}.$$

Thus

$$D_{(y,z)}F = \begin{bmatrix} x \cos y & e^{x+z} \\ x^2 + z e^{yz} & y e^{yz} - 2z \end{bmatrix}, \qquad D_{(y,z)}(1, 0, -1) = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}.$$

Since

$$\det D_{(y,z)}(1, 0, -1) = 1(-2) - 1(0) = -2 \neq 0,$$

the matrix $D_{(y,z)}(1, 0, -1)$ is invertible. Thus, the implicit function theorem applies, and we can solve for (y, z) in terms of x near the point $(1, 0, -1)$.

However,

$$D_{(y,z)}(-1, 0, 1) = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix},$$

which is not invertible. Thus, the implicit function theorem does not apply. On the other hand,

$$D_{(x,z)} = \begin{bmatrix} \sin y + e^{x+z} & e^{x+z} \\ 2xy & y e^{yz} - 2z \end{bmatrix}, \quad \text{and} \quad D_{(x,z)}(-1, 0, 1) = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

is invertible. So the implicit function theorem tells us that we can solve for (x, z) in terms of y near the point $(-1, 0, 1)$. △

Warning 1.11.8. The converses to Theorems 1.11.2 and 1.11.5 are false! For example, if $\frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) = 0$ in the setting of Theorem 1.11.2, this does *not* imply that it impossible to write z as a function of $\mathbf{x}$ near $(\mathbf{x}_0, z_0)$. See Exercise 1.11.1.

Exercises.

1.11.1. Use the function $F(x,z) = x^3 - z^3$ to show that the converse to Theorem 1.11.2 is false.

1.11.2 ([MT11, §3.5, Exercise 1]). Show that the equation $x + y - z + \cos(xyz) = 1$ can be solved for $z = g(x,y)$ near the origin. Find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ at $(0,0)$.

1.11.3 ([MT11, §3.5, Exercise 2]). Show that $xy + z + 3xz^5 = 4$ is solvable for z as a function of (x,y) near $(1,0,1)$. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x,y) = (1,0)$.

1.11.4 ([MT11, §3.5, Exercise 6]). Consider the surface S given by $3y^2z^2 - 3x = 0$.

(a) Using the implicit function theorem, verify that we can solve for x as a function of y and z near any point on S. Explicitly write x as a function of y and z.

(b) Show that near $(1,1,-1)$ we can solve for either y or z, and give explicit expressions for these variables in terms of the other two.

1.11.5 ([MT11, §3.5, Exercise 7]). Show that $x^3z^2 - z^3yx = 0$ is solvable for z as a function of (x,y) near $(1,1,1)$, but not near the origin. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x,y) = (1,1)$.

1.11.6 ([MT11, §3.5, Exercise 8]). Discuss the solvability of the system

$$\begin{aligned} 3x + 2y + z^2 + u + v^2 &= 0 \\ 4x + 3y + z + u^2 + v + w + 2 &= 0 \\ x + z + w + u^2 + 2 &= 0 \end{aligned}$$

for u, v, w in terms of x, y, z near $x = y = z = 0$, $u = v = 0$, and $w = -2$.

1.11.7 ([MT11, §3.5, Exercise 14]). Consider the unit sphere

$$S = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}.$$

This sphere intersects the x-axis at two points. Which variables can we solve for at these points? What about the points of intersection of S with the y-axis and z-axis?

1.12 The inverse function theorem

A special case of the implicit function theorem (Theorem 1.11.5) is the *inverse function theorem.* Suppose we have a function

$$f\colon \mathbb{R}^n \to \mathbb{R}^n.$$

When can we invert this function, at least locally (that is, in a neighbourhood of a point)? The inverse function theorem says that we can do this when the derivative is invertible at that point.

Theorem 1.12.1 (Inverse function theorem)**.** *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-function, $\mathbf{x}_0 \in \mathbb{R}^n$, and $Df(\mathbf{x}_0)$ is an invertible matrix. Then there exists a neighbourhood U of $\mathbf{x}_0$ and a neighbourhood V of $f(\mathbf{x}_0)$ such that $f|_U\colon U \to V$ is a bijection (hence invertible). Moreover, $(f|_U)^{-1}\colon V \to U$ is C^1*

Proof. Consider the function

$$F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad F(\mathbf{x},\mathbf{y}) = f(\mathbf{x}) - \mathbf{y}.$$

Thus,

$$\mathbf{y} = f(\mathbf{x}) \iff F(\mathbf{x},\mathbf{y}) = \mathbf{0}.$$

Let $\mathbf{y}_0 = f(\mathbf{x}_0)$, so that $F(\mathbf{x}_0,\mathbf{y}_0) = \mathbf{0}$.

Since

$$D_{\mathbf{x}}F = Df,$$

our assumption that $Df(\mathbf{x}_0)$ is invertible implies that $D_{\mathbf{x}}F(\mathbf{x}_0)$ is invertible. Hence, the implicit function theorem (Theorem 1.11.5) implies that there exist neighbourhoods V of $\mathbf{y}_0$ and U of $\mathbf{x}_0$ such that there is a unique function

$$g\colon V \to U \quad \text{such that} \quad F(g(\mathbf{y}),\mathbf{y}) = \mathbf{0} \text{ for all } \mathbf{y} \in V.$$

In other words, g is the unique function satisfying

$$\mathbf{y} = f(g(\mathbf{y})) \text{ for all } \mathbf{y} \in V. \tag{1.21}$$

Thus $g = (f|_U)^{-1}$. (More precisely, (1.21) shows that g is a right inverse to $f|_U$, implying that $f|_U$ is surjective. The uniqueness of g shows that f is also injective, so that g is a two-sided inverse to $f|_U$.) □

Example 1.12.2 (Spherical coordinates)*.* Consider the function

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad f(\rho,\theta,\varphi) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi).$$

Then we have

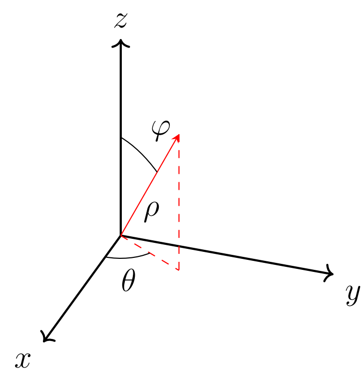
$$Df = \begin{bmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix}.$$

Using some basic trigonometric formulas (Exercise 1.12.4), we can compute that

$$\det(Df) = -\rho^2 \sin\varphi. \tag{1.22}$$

Therefore, we can apply the invertible function theorem (Theorem 1.12.1) as long as $\rho \neq 0$ and $\sin\varphi \neq 0$ (that is, φ is not a multiple of π).

The function f maps (ρ, θ, φ) to the point $(x, y, z) \in \mathbb{R}^3$ whose spherical coordinates are (ρ, θ, φ):



When $\rho = 0$, we are at the origin. Then changes in θ and φ do not change the point (x, y, z). Thus, the function is not locally invertible when $\rho = 0$. Similarly, when φ is a multiple of π, we are on the z-axis, and changes in θ do not change the point. We will discuss spherical coordinates in more detail in Section 4.5. △

Warning 1.12.3. The converse to Theorem 1.12.1 is false! See Exercise 1.12.1.

Exercises.

1.12.1. Show that the converse to Theorem 1.12.1 is false by giving a counterexample. *Hint*: Try $n = 1$ and $f(x) = x^3$.

1.12.2 ([MT11, §3.5, Exercise 11]). Consider

$$f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2, \quad f(x,y) = \left(\frac{x^2 - y^2}{x^2 + y^2}, \frac{xy}{x^2 + y^2} \right).$$

Does this map have a local inverse near $(x, y) = (0, 1)$?

1.12.3 ([MT11, §3.5, Exercise 12]). (a) Define

$$F(r,\theta) = \big(x(r,\theta), y(r,\theta)\big) = (r\cos\theta, r\sin\theta).$$

Show that the determinant of the Jacobian matrix at (r_0, θ_0) is

$$\det DF(r_0, \theta_0) = r_0.$$

(b) When does there exist a smooth inverse function $(r(x, y), \theta(x, y))$? Check directly and with the inverse function theorem.

1.12.4. Prove (1.22).

Chapter 2

Extrema

One of the important applications of single-variable calculus that you saw in previous courses is finding maxima and minima of functions of one variable. We would now like to extend these techniques to functions of several variables. A good reference for the material in this chapter is [FRYc, §2.9, §2.10].

2.1 Maximum and minimum values

We begin with some definitions. Recall the definition of the open ball $B_r(\mathbf{a})$ given in Definition 1.1.1.

Definition 2.1.1 (Maximum and minimum)**.** Suppose $A \subseteq \mathbb{R}^n$, $f\colon A \to \mathbb{R}$, and $\mathbf{a} \in A$.

- We say $\mathbf{a}$ is a *local maximum* of f if there exists $r > 0$ such that

$$f(\mathbf{x}) \le f(\mathbf{a}) \quad \text{for all } \mathbf{x} \in B_r(\mathbf{a}).$$

- We say $\mathbf{a}$ is a *local minimum* of f if there exists $r > 0$ such that

$$f(\mathbf{x}) \ge f(\mathbf{a}) \quad \text{for all } \mathbf{x} \in B_r(\mathbf{a}).$$

- We say $\mathbf{a}$ is a *global maximum*, or *absolute maximum*, of f if

$$f(\mathbf{x}) \le f(\mathbf{a}) \quad \text{for all } \mathbf{x} \in A.$$

- We say $\mathbf{a}$ is a *global minimum*, or *absolute minimum* of f if

$$f(\mathbf{x}) \ge f(\mathbf{a}) \quad \text{for all } \mathbf{x} \in A.$$

We say $\mathbf{a}$ is a *local extremum* if it is a local minimum or a local maximum. We say $\mathbf{a}$ is a *global extremum* if it is a global minimum or a global maximum. The plurals of maximum, minimum, and extremum, are *maxima*, *minima*, and *extrema*, respectively.

Definition 2.1.2 (Critical point)**.** Suppose $A \subseteq \mathbb{R}^n$ and $f\colon A \to \mathbb{R}$. We say $\mathbf{a} \in A$ is a *critical point* if either $\nabla f = \mathbf{0}$ or f is not differentiable at $\mathbf{a}$.

Remark 2.1.3. If $\mathbf{a} \in \partial A$ (that is, $\mathbf{a}$ is a boundary point of A), then f is not differentiable at $\mathbf{a}$. Hence, all boundary points are critical points.

Remark 2.1.4. Some texts (e.g. [FRYc]) use the term *critical point* only to refer to points $\mathbf{a}$ such that $\nabla f(\mathbf{a}) = \mathbf{0}$, and call $\mathbf{a} \in A$ a *singular point* if f is not differentiable at $\mathbf{a}$.

You learned in single-variable calculus that every continuous function defined on a closed interval attains a global maximum and minimum. This result can be generalized to multiple variables.

Definition 2.1.5 (Bounded, compact)**.** We say that the set $A \subseteq \mathbb{R}^n$ is *bounded* if there exists some $r > 0$ such that $A \subseteq B_r(\mathbf{0})$. If A is bounded and contains its boundary (that is, $\partial A \subseteq A$), we say A is *compact*.

Examples 2.1.6.

(a) For $r > 0$ and $\mathbf{a} \in \mathbb{R}^n$, the open ball $B_r(\mathbf{a})$ is not compact since it does not contain its boundary.

(b) For $r > 0$ and $\mathbf{a} \in \mathbb{R}^n$, the closed ball $\{\mathbf{x} : \|\mathbf{x} - \mathbf{a}\| \leq r\}$ is compact.

(c) The sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ is compact since it is bounded and has no boundary (hence it contains its boundary, since the empty set is contained in every set).

(d) The set $[0, 1]^2$ is compact.

(e) The set $A = [0, 1] \times (0, 1]$ is not compact, since it does not contain its boundary. For example, $(0.5, 0)$ is a boundary point of A that is not contained in A.

(f) The set $A = \mathbb{R} \times [0, 1]$ is not compact, since is it not bounded.

△

Theorem 2.1.7 (Extremum value theorem)**.** *Suppose $A \subseteq \mathbb{R}^n$ is compact, and let $f \colon A \to \mathbb{R}$ be continuous. Then f attains a global maximum and minimum. In other words, there exist $\mathbf{x}_{\min}, \mathbf{x}_{\max} \in A$ such that*

$$f(\mathbf{x}_{\min}) \leq f(\mathbf{x}) \leq f(\mathbf{x}_{\max}) \quad \textit{for all } \mathbf{x} \in A.$$

Proof. The proof of this theorem uses concepts from real analysis, and is beyond the scope of this course. For an idea of the proof, see http://www.cut-the-knot.org/fta/fta_note.shtml. □

Example 2.1.8. For $a, b \in \mathbb{R}$ with $a < b$. The interval $[a, b] \subseteq \mathbb{R}^2$ is bounded and contains its boundary, hence is compact. Then Theorem 2.1.7 reduces to the extreme value theorem you saw in single variable calculus: any continuous function $f \colon [a, b] \to \mathbb{R}$ attains a global maximum and minimum. △

A function $f \colon A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$ is *bounded* if there is a number $M > 0$ such that

$$-M \leq f(x, y) \leq M \quad \text{for all } (x, y) \in A.$$

Note that this is equivalent to the image $f(A)$ of f being a bounded set in the sense of Definition 2.1.5.

Lemma 2.1.9. *If $A \subseteq \mathbb{R}^n$ is compact and $f\colon A \to \mathbb{R}$ is continuous, then f is bounded.*

Proof. By Theorem 2.1.7, f attains a global maximum M_1 and a global minimum m_1. If we define $M = \max\{|M_1|, |m_1|\}$, then for all $(x, y) \in A$, we have

$$-M \leq m_1 \leq f(x, y) \leq M_1 \leq M.$$

Hence f is bounded. □

Theorem 2.1.10 (First derivative test)**.** *Every local extremum is a critical point.*

Proof. Suppose $A \subseteq \mathbb{R}^n$ and $f\colon A \to \mathbb{R}$. Suppose $\mathbf{a} = (a_1, \ldots, a_n) \in A$ is a local extremum. If f is not differentiable at $\mathbf{a}$, then $\mathbf{a}$ is a critical point, and we are done. Thus, we suppose f is differentiable at $\mathbf{a}$. For $1 \leq i \leq n$, consider the function

$$g_i\colon \mathbb{R} \to \mathbb{R}, \quad g_i(x) = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n).$$

Since $\mathbf{a}$ is a local extremum for f, it is also a local extremum for g_i. By the single-variable first derivative test, we have

$$0 = \frac{\mathrm{d}g_i}{\mathrm{d}x}(a_i) = \frac{\partial f}{\partial x_i}(\mathbf{a}).$$

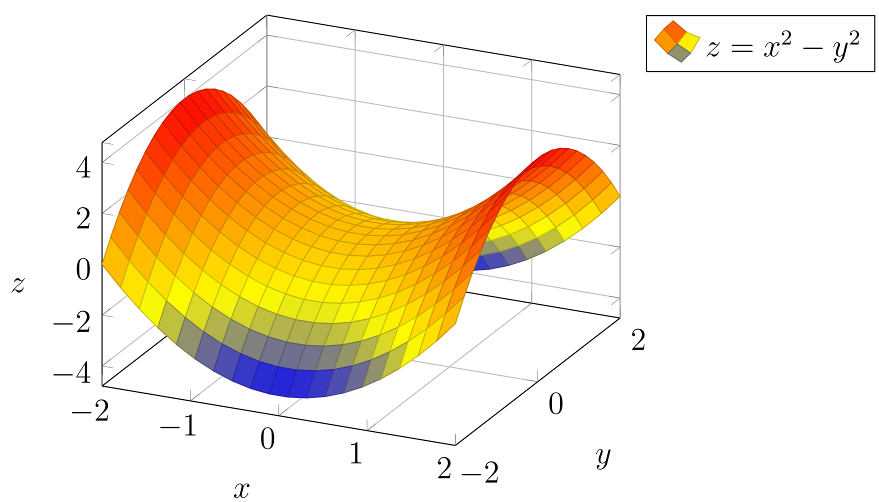
Hence $\nabla f(\mathbf{a}) = \mathbf{0}$. □

Warning 2.1.11. The converse to Theorem 2.1.10 is false, as the following example illustrates.

Example 2.1.12. Consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^2 - y^2,$$

Its graphs looks as follows:



We have $\nabla f = (2x, -2y)$. Hence $\nabla f(0, 0) = \mathbf{0}$, and so $(0, 0)$ is a critical point. However, $(0, 0)$ is not a local extremum. We have $f(0, 0) = 0$. However, for any $r > 0$, the ball $B_r(0, 0)$ contains the points $\left(\frac{r}{2}, 0\right)$ and $\left(0, \frac{r}{2}\right)$. Since

$$f\left(\tfrac{r}{2}, 0\right) > 0 \qquad \text{and} \qquad f\left(0, \tfrac{r}{2}\right) < 0,$$

the point $(0, 0)$ is neither a local maximum nor a local minimum. △

Definition 2.1.13 (Saddle point)**.** Suppose $A \subseteq \mathbb{R}^n$, $f \colon A \to \mathbb{R}$, and $\mathbf{a} \in A$. We say that $\mathbf{a}$ (or the corresponding point $(\mathbf{a}, f(\mathbf{a}))$ on the graph of f) is a *saddle point* if $\nabla f(\mathbf{a}) = \mathbf{0}$, but $\mathbf{a}$ is not a local extremum of f.

In Example 2.1.12, $(0,0,0)$ is saddle point for f.

Example 2.1.14. Let us find the critical points of the function

$$f \colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = x^2y + xy^2 + xy.$$

We have

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2xy + y^2 + y = y(2x+y+1),\\ \frac{\partial f}{\partial y} &= x^2 + 2xy + x = x(x+2y+1).\end{aligned}$$

Since f is differentiable everywhere, we must solve $\nabla f(x,y) = (0,0)$. So we have

$$y(2x+y+1) = 0 \qquad \text{and} \qquad x(x+2y+1) = 0.$$

Thus

$$(y = 0 \text{ or } 2x+y+1 = 0) \quad \text{and} \quad (x = 0 \text{ or } x+2y+1 = 0).$$

So we have four cases:

- $x = 0$ and $y = 0$.
- $y = 0$ and $x + 2y + 1 = 0$, which implies that $x = -1$.
- $2x + y + 1 = 0$ and $x = 0$, which implies that $y = -1$.
- $2x + y + 1 = 0$ and $x + 2y + 1 = 0$. Solving this linear system (exercise!) gives $x = y = -\frac{1}{3}$.

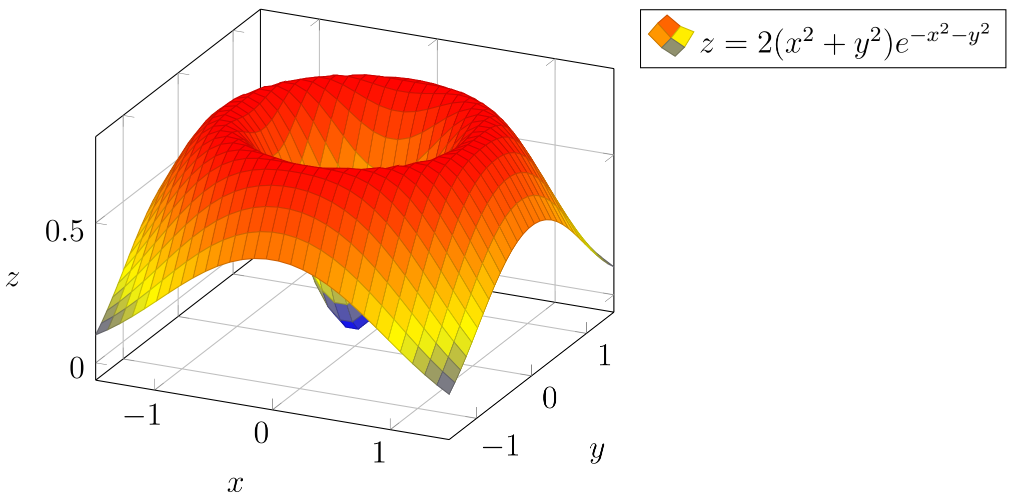
Thus, there are four critical points:

$$(0,0), \quad (-1,0), \quad (0,-1), \quad \left(-\tfrac{1}{3}, -\tfrac{1}{3}\right).$$

If f has any local extrema, then they must be amongst these points. We will return to this example in Example 2.2.5. $\triangle$

Exercises.

2.1.1. Consider the function $f(x, y) = 2(x^2 + y^2)e^{-x^2-y^2}$, whose graph looks as follows:



Find the critical points.

Exercises from [FRYb, §2.9]: Q1–Q3

2.2 The second derivative test

If we are searching for extrema, the first derivative test (Theorem 2.1.10) allows us to restrict our attention to critical points. However, once we have found the critical points, how do we determine if they are extrema? One useful tool is the *second derivative test*, which we now explore. We first need a few facts from linear algebra.

Note that, if $B \in \operatorname{Mat}_{n\times n}(\mathbb{R})$ and $\mathbf{h} \in \mathbb{R}^n$, then $\mathbf{h}^\intercal B\mathbf{h}$ is a 1×1 matrix, which we sometimes just view as a real number.

Definition 2.2.1 (Positive/negative definite matrix)**.** A square matrix $B \in \operatorname{Mat}_{n\times n}(\mathbb{R})$ is *positive definite* if

$$\mathbf{h}^\intercal B\mathbf{h} > 0 \quad \text{for all } \mathbf{h} \in \mathbb{R}^n,\ \mathbf{h} \neq \mathbf{0}.$$

The matrix B is *negative definite* if

$$\mathbf{h}^\intercal B\mathbf{h} < 0 \quad \text{for all } \mathbf{h} \in \mathbb{R}^n,\ \mathbf{h} \neq \mathbf{0}.$$

(Note that a matrix can be neither positive definite nor negative definite.)

For a matrix

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \in \operatorname{Mat}_{n\times n}(\mathbb{R}),$$

define the submatrices

$$B^{(1)} = \begin{bmatrix} b_{11} \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \dots, \quad B^{(n)} = B.$$

In other words, for $1 \le m \le n$, $B^{(m)}$ is the top-left $m \times m$ submatrix of B. Recall that the matrix B is *symmetric* if $b_{ij} = b_{ji}$ for $1 \le i, j \le m$. For symmetric matrices, there is an easy way to determine if the matrix is positive or negative definite.

Proposition 2.2.2. *Suppose $B \in \operatorname{Mat}_{n\times n}(\mathbb{R})$ is symmetric.*

(a) *The matrix B is positive definite if and only if*

$$\det(B^{(m)}) > 0 \quad \text{for all } 1 \le m \le n.$$

(b) *The matrix B is negative definite if and only if, for $1 \le m \le n$,*

$$\det(B^{(m)}) < 0 \quad \text{if } m \text{ is odd}$$

and

$$\det(B^{(m)}) > 0 \quad \text{if } m \text{ is even.}$$

Proof. For a proof of this result, see [Nic, Th. 8.3.3]. □

Recall the Hessian matrix from (1.18)

$$\mathbf{H}f(\mathbf{a}) := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_1\,\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n}(\mathbf{a}) \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_n\,\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{a}) \end{bmatrix} \in \operatorname{Mat}_{n\times n}(\mathbb{R}).$$

If f is of class C^2 at $\mathbf{a}$, then it follows from Clairaut's theorem (Theorem 1.9.6) that $\mathbf{H}f(\mathbf{a})$ is a symmetric matrix.

Theorem 2.2.3 (Second derivative test)**.** *Suppose $A \subseteq \mathbb{R}^n$ is open, $\mathbf{x}_0 \in A$, $f\colon A \to \mathbb{R}$ is of class C^2, and $\nabla f(\mathbf{x}_0) = \mathbf{0}$.*

(a) *If $\mathbf{H}f(\mathbf{x}_0)$ is positive definite, then $\mathbf{x}_0$ is a local minimum.*

(b) *If $\mathbf{H}f(\mathbf{x}_0)$ is negative definite, then $\mathbf{x}_0$ is a local maximum.*

(c) *If $\det(\mathbf{H}f(\mathbf{x}_0)) \ne 0$ and $\mathbf{H}f(\mathbf{x}_0)$ is neither positive definite nor negative definite, then $\mathbf{x}_0$ is a saddle point.*

(d) *If $\det(\mathbf{H}f(\mathbf{x}_0)) = 0$ then we cannot draw any conclusion without additional work.*

Proof. For a detailed proof, see [MT11, §3.3, Th. 5]. For a general idea, note that, when $\nabla f(\mathbf{x}_0) = \mathbf{0}$, the degree 2 Taylor polynomial from (1.19) becomes

$$p_2(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \frac{1}{2}\mathbf{h}^\intercal(\mathbf{H}f(\mathbf{x}_0))\mathbf{h}.$$

If $\mathbf{H}f(\mathbf{x}_0)$ is positive definite, then $p_2(\mathbf{x}_0 + \mathbf{h}) > f(\mathbf{x}_0)$ for $\mathbf{h} \ne \mathbf{0}$. Since p_2 is a good approximation to f near $\mathbf{x}_0$, this implies that $f(\mathbf{x}) \ge f(\mathbf{x}_0)$ for $\mathbf{x}$ near $\mathbf{x}_0$, and hence $\mathbf{x}_0$ is a local minimum. Similarly, if $\mathbf{H}_f(\mathbf{x}_0)$ is negative definite, then $f(\mathbf{x}) \le f(\mathbf{x}_0)$ for $\mathbf{x}$ near $\mathbf{x}_0$, and so $\mathbf{x}_0$ is a local maximum. □

It can be useful for reference purposes to give the $n = 2$ case of the second derivative test explicitly.

Corollary 2.2.4 (Two-variable second derivative test)**.** *Suppose $A \subseteq \mathbb{R}^2$ is open, $(a,b) \in A$, $f\colon A \to \mathbb{R}$ is of class C^2, and $\nabla f(a,b) = \mathbf{0}$. Then we have*

$$\det(\mathbf{H}f(a,b)) = \frac{\partial^2 f}{\partial x^2}(a,b)\frac{\partial^2 f}{\partial y^2}(a,b) - \left(\frac{\partial^2 f}{\partial x\,\partial y}(a,b)\right)^2.$$

(a) *If*
$$\frac{\partial^2 f}{\partial x^2}(a,b) > 0 \quad \textit{and} \quad \det(\mathbf{H}f(a,b)) > 0$$
then (a,b) is a local minimum.

(b) *If*
$$\frac{\partial^2 f}{\partial x^2}(a,b) < 0 \quad \textit{and} \quad \det(\mathbf{H}f(a,b)) > 0$$
then (a,b) is a local maximum.

(c) *If*
$$\det(\mathbf{H}f(a,b)) < 0$$
then (a,b) is a saddle point.

(d) *If*
$$\det(\mathbf{H}f(a,b)) = 0$$
then we cannot draw any conclusion without additional work.

Example 2.2.5. Consider again the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = x^2y + xy^2 + xy.$$

In Example 2.1.14, we found that the critical points of f are

$$(0,0), \quad (-1,0), \quad (0,-1), \quad \left(-\tfrac{1}{3}, -\tfrac{1}{3}\right).$$

The second partial derivatives of f are

$$\frac{\partial^2 f}{\partial x^2} = 2y, \qquad \frac{\partial^2 f}{\partial x\,\partial y} = 2x + 2y + 1, \qquad \frac{\partial^2 f}{\partial y^2} = 2x.$$

Therefore, the Hessian matrix is

$$\mathbf{H}f = \begin{bmatrix} 2y & 2x+2y+1 \\ 2x+2y+1 & 2x \end{bmatrix},$$

and so

$$\mathbf{H}f(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{H}f(-1,0) = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix},$$
$$\mathbf{H}f(0,-1) = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{H}f\left(-\tfrac{1}{3}, -\tfrac{1}{3}\right) = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}.$$

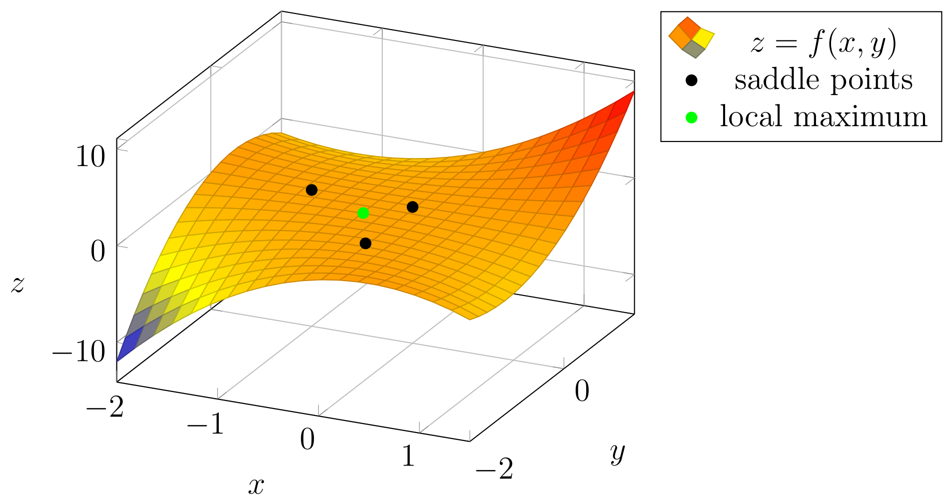
We can then look at the critical points:

- Since $\det(\mathbf{H}f(0,0)) = -1 < 0$, the point $(0,0,0)$ is a saddle point.
- Since $\det(\mathbf{H}f(-1,0)) = -1 < 0$, the point $(-1,0,0)$ is a saddle point.
- Since $\det(\mathbf{H}f(0,-1)) = -1 < 0$, the point $(0,-1,0)$ is a saddle point.
- Since

  $$\frac{\partial^2 f}{\partial x^2}\left(-\tfrac{1}{3}, -\tfrac{1}{3}\right) = -\tfrac{2}{3} < 0 \quad \text{and} \quad \det\left(\mathbf{H}f\left(-\tfrac{1}{3}, -\tfrac{1}{3}\right)\right) = \left(-\tfrac{2}{3}\right)^2 - \left(-\tfrac{1}{3}\right)^2 = \tfrac{1}{3} > 0,$$

  the function attains a local maximum at $\left(-\tfrac{1}{3}, -\tfrac{1}{3}\right)$.

In fact, the graph of f looks as follows:



We see that our local maximum is *not* a global maximum. △

Exercises.

Exercises from [FRYb, §2.9]: Q4–Q29

2.3 Constrained extrema

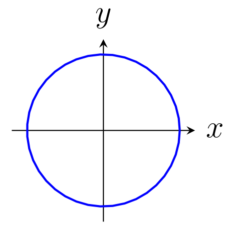
We often want to maximize or minimize a function subject to some *constraint*. Geometrically, this constraint means that we want to restrict the domain of our function.

Example 2.3.1. Consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = \sqrt{3}x - y.$$

We want to find the maximum or minimum value of this function, but not on its entire domain $\mathbb{R}^2$. Rather, we want to find the maximum value on the unit circle:

$$S = \{(x, y) : x^2 + y^2 = 1\},$$



One way to solve this problem would be to parameterize the circle, compose f with the parameterization, then find the extrema as usual. Since

$$S = \{(\cos t, \sin t) : t \in [0, 2\pi]\},$$

we can find the extrema of

$$h(t) := f(\cos t, \sin t) = \sqrt{3}\cos t - \sin t.$$

This is now a single-variable calculus problem. We solve

$$0 = h'(t) = -\sqrt{3}\sin t - \cos t \implies \tan t = \frac{\sin t}{\cos t} = -\frac{1}{\sqrt{3}} \implies t = -\frac{\pi}{6} \text{ or } t = \frac{5\pi}{6}.$$

At $t = -\frac{\pi}{6}$, we have

$$\cos t = \frac{\sqrt{3}}{2}, \ \sin t = -\frac{1}{2}, \quad f(\cos t, \sin t) = \frac{3}{2} + \frac{1}{2} = 2.$$

At $t = \frac{5\pi}{6}$, we have

$$\cos t = -\frac{\sqrt{3}}{2}, \ \sin t = \frac{1}{2}, \quad f(\cos t, \sin t) = -\frac{3}{2} - \frac{1}{2} = -2.$$

Since the domain of h is $[0, 2\pi]$, which is a closed and bounded interval (hence compact), we know that h attains a maximum and minimum (by the extreme value theorem). Therefore, the maximum value of f on the unit circle is 2, at the point $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, and the minimum value of f on the unit circle is -2, at the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. $\triangle$

In theory, we can take a similar approach to finding extrema of functions restricted to surfaces as well. In that case, our parameterization will involve two variables, and we would reduce the problem to finding extreme for a two-variable function.

However, there are some limitations to the above approach:

- It is not always easy to find a parameterization of a constraint.
- This approach can sometimes get messy, especially if the parameterization is complicated.
- This approach is not as elegant as the Lagrange multiplier method we will develop in the next section.

2.4 Lagrange multipliers

A good reference for the material in this section is [FRYc, §2.10].

Recall that if $f\colon A \to B$ is a function, and $S \subseteq A$, then

$$f|_S\colon S \to B, \quad f|_S(\mathbf{x}) := f(\mathbf{x}),$$

is the *restriction* of f to S. Finding extrema for $f|_S$ is the same as finding extrema for f subject to the constraint that points lie in S. For example $\mathbf{a} \in S$ is a local minimum for $f|_S$ if there exists $r > 0$ such that

$$f(\mathbf{a}) \le f(\mathbf{x}) \quad \text{for all } \mathbf{x} \in B_r(\mathbf{a}) \cap S.$$

Theorem 2.4.1 (Lagrange multipliers)**.** *Suppose $U \subseteq \mathbb{R}^n$ is open and $f, g\colon U \to \mathbb{R}$ are C^1 functions. Furthermore, suppose $k \in \mathbb{R}$ and*

$$S = \{\mathbf{x} \in U : g(\mathbf{x}) = k\}.$$

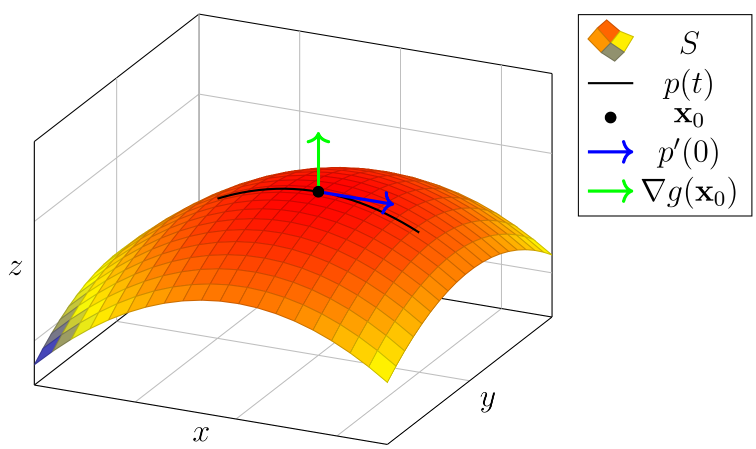
If $\mathbf{x}_0 \in S$ is a local extremum for $f|_S$, then $\nabla f(\mathbf{x}_0)$ and $\nabla g(\mathbf{x}_0)$ are linearly dependent, that is, either

- $\nabla g(\mathbf{x}_0) = 0$, *or*
- $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$ *for some $\lambda \in \mathbb{R}$ (possibly $\lambda = 0$). The number λ is a called a* Lagrange multiplier*.*

We call a point $\mathbf{x}_0$ where $\nabla f(\mathbf{x}_0)$ and $\nabla g(\mathbf{x}_0)$ are linearly dependent a *critical point.*

Proof. Let $\mathbf{x}_0 \in S$ be a local extremum for $f|_S$. Suppose we have a C^1 path in S passing through $\mathbf{x}_0$. More precisely, suppose we have a C^1 function

$$p\colon [-1, 1] \to S, \quad p(0) = \mathbf{x}_0.$$



Then the function

$$f \circ p\colon [-1, 1] \to \mathbb{R}$$

has a local extremum at 0. Since it is of class C^1, this implies that its derivative is zero. Using the chain rule (Theorem 1.6.2), we have

$$0 = (f \circ p)'(\mathbf{x}_0) = \nabla f(p(0)) \cdot p'(0) = \nabla f(\mathbf{x}_0) \cdot p'(0) \implies \nabla f(\mathbf{x}_0) \perp p'(0).$$

Since this is true for *all* paths in S that pass through $\mathbf{x}_0$ at time 0, we see that $\nabla f(\mathbf{x}_0)$ is orthogonal to *all* tangent vectors to S at $\mathbf{x}_0$. Thus, $\nabla f(\mathbf{x}_0)$ is orthogonal to S at $\mathbf{x}_0$.

But we already know, by Proposition 1.8.6, that $\nabla g(\mathbf{x}_0)$ is orthogonal to S at $\mathbf{x}_0$. Therefore, $\nabla f(\mathbf{x}_0)$ and $\nabla g(\mathbf{x}_0)$ are parallel vectors. In other words, they are linearly dependent. □

Example 2.4.2. Let's return to Example 2.3.1 and find the extrema using the method of Lagrange multipliers. We define

$$g(x, y) = x^2 + y^2,$$

and we want to find the extrema of

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \sqrt{3}x - y,$$

subject to the constraint

$$g(x, y) = 1.$$

We compute

$$\nabla f = (\sqrt{3}, -1), \qquad \nabla g = (2x, 2y).$$

Since ∇g is only zero at $(0, 0)$, which does not satisfy the constraint $g(x, y) = 1$, we want to solve the vector equation $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$. Thus, we want to solve

$$(\sqrt{3}, -1) = \lambda(2x, 2y).$$

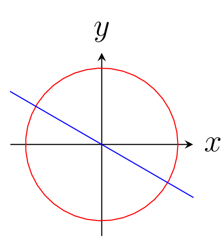
So we have three equations in three variables (don't forget the constraint!):

$$2\lambda x = \sqrt{3}, \qquad 2\lambda y = -1, \qquad x^2 + y^2 = 1.$$

It follows from the first two equations that $\lambda \neq 0$. Thus, we can solve these first two equations for x and y, giving

$$(x, y) = \left(\tfrac{\sqrt{3}}{2\lambda}, -\tfrac{1}{2\lambda} \right) = \tfrac{1}{\lambda} \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2} \right). \tag{2.1}$$

Note that this is the parametric equation of a line through the origin (with the origin removed, since $1/\lambda$ can never be zero). We want to find the points where this line intersects the unit circle.



To find these points of intersection, we substitute the expression for x and y from (2.1) into the constraint, to get

$$1 = \left(\frac{\sqrt{3}}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 \implies 4\lambda^2 = 3 + 1 \implies \lambda^2 = 1 \implies \lambda = \pm 1.$$

Substituting these two values for λ into (2.1) yields the points

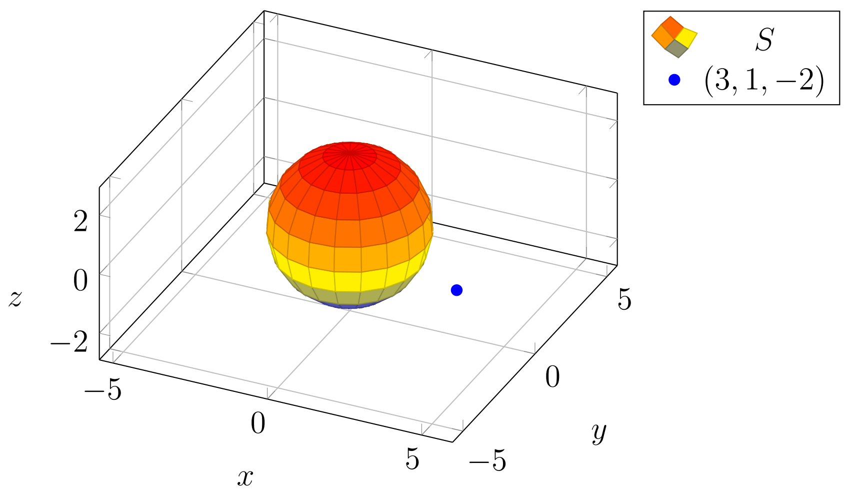
$$(x, y) = \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right) \qquad \text{and} \qquad (x, y) = \left(-\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right)$$

These are the same points we found in Example 2.3.1, where we then concluded that the maximum value of f on the unit circle is 2, at the point $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, and the minimum value of f on the unit circle is -2, at the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. △

Example 2.4.3. Let's find the point on the sphere

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 6\}$$

that is closest to the point $(3, 1, -2)$.



Instead of minimizing the distance, we will minimize the square of the distance from $(3, 1, -2)$, since this avoids some nasty square roots. (Since the square function $t \mapsto t^2$ is increasing, the square of the distance attains its minimum precisely where the distance attains its minimum.) So we want to minimize the function

$$f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 2)^2$$

subject to the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 = 6.$$

We have

$$\nabla f(x, y, z) = \big(2(x - 3), 2(y - 1), 2(z + 2)\big), \quad \nabla g(x, y, z) = (2x, 2y, 2z).$$

So we need to solve the equations

$$
\begin{aligned}
2(x-3) &= 2\lambda x & &\iff & x &= \frac{3}{1-\lambda} \qquad (2.2)\\
2(y-1) &= 2\lambda y & &\iff & y &= \frac{1}{1-\lambda} \qquad (2.3)\\
2(z+2) &= 2\lambda z & &\iff & z &= -\frac{2}{1-\lambda} \qquad (2.4)\\
x^2+y^2+z^2 &= 6. & & & & \qquad (2.5)
\end{aligned}
$$

(Note that, in the first equation, $\lambda = 1$ yields the contradiction $-6 = 0$. So we can assume $\lambda \neq 1$, allowing us to divide by $1 - \lambda$.) Substituting (2.2) to (2.4) into (2.5) gives

$$\frac{14}{(1-\lambda)^2} = 6 \implies 6(1-\lambda)^2 = 14 \implies 3\lambda^2 - 6\lambda - 4 = 0 \implies \lambda = 1 \pm \frac{\sqrt{21}}{3},$$

where we used the quadratic formula in the last step. Substituting these values of λ into (2.2) to (2.4) gives

$$(x,y,z) = \left(\mp\frac{9}{\sqrt{21}}, \mp\frac{3}{\sqrt{21}}, \pm\frac{6}{\sqrt{21}}\right).$$

The point $\left(\frac{9}{\sqrt{21}}, \frac{3}{\sqrt{21}}, -\frac{6}{\sqrt{21}}\right)$ is clearly closer to $(3, 1, -2)$ than the point $\left(-\frac{9}{\sqrt{21}}, -\frac{3}{\sqrt{21}}, \frac{6}{\sqrt{21}}\right)$ is. So the nearest point is $\left(\frac{9}{\sqrt{21}}, \frac{3}{\sqrt{21}}, -\frac{6}{\sqrt{21}}\right)$ and the farthest point is $\left(-\frac{9}{\sqrt{21}}, -\frac{3}{\sqrt{21}}, \frac{6}{\sqrt{21}}\right)$. △

Exercises.

Exercises from [FRYb, §2.10]: Q1–Q9, Q12, Q13, Q15–25, Q27, Q28.

2.5 Multiple constraints

In Section 2.4, we learned how to use the method of Lagrange multipliers to maximize or minimize a function subject to *one* constraint. However, sometimes we want to maximize or minimize a function subject to *several* constraints. It turns out that the method of Lagrange multipliers can be generalized to accommodate this scenario. For a reference discussing two constraints, see [FRYc, §2.10.1].

Recall that vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are *linearly dependent* if there exist scalars $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, not all zero, such that

$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k = \mathbf{0}.$$

Equivalently, the vectors are linearly dependent if at least one of them can be written as a linear combination of the others.

Theorem 2.5.1 (Lagrange multipliers, several constraints)**.** *Suppose $U \subseteq \mathbb{R}^n$ is open,*

$$f, g_1, \dots, g_r \colon U \to \mathbb{R}$$

are C^1 functions, and $k_1, \dots, k_r \in \mathbb{R}$. Let

$$S = \{\mathbf{x} \in U : g_i(\mathbf{x}) = k_i \text{ for all } i = 1, \dots, r\}.$$

If $\mathbf{x}_0 \in S$ is a local extrema for $f|_S$, then the vectors

$$\nabla f(\mathbf{x}_0), \nabla g_1(\mathbf{x}_0), \dots, \nabla g_r(\mathbf{x}_0)$$

are linearly dependent.

Example 2.5.2. Let's find the extrema of

$$f \colon \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z) = x - y,$$

subject to the constraints

$$g_1(x, y, z) = x^2 + y^2 + z^2 = 11 \quad \text{and} \quad g_2(x, y, z) = x + y + z = 3.$$

One can show that the set

$$S = \{(x, y, z) : g_1(x, y, z) = 11 \text{ and } g_2(x, y, z) = 3\}$$

contains its boundary. Thus, by the extreme value theorem (Theorem 2.1.7) $f|_S$ attains a maximum and a minimum. So we can use the method of Lagrange multipliers to find these extrema.

We compute

$$\nabla f = (1, -1, 0), \quad \nabla g_1 = (2x, 2y, 2z), \quad \nabla g_2 = (1, 1, 1).$$

Since ∇f and ∇g_2 are clearly linearly independent[1], these three vectors are linearly dependent if and only if ∇g_1 can be written as a linear combination of ∇f and ∇g_2. So we have

$$\nabla g_1 = \lambda_1 \nabla f + \lambda_2 \nabla g_2$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Therefore, we want to solve the following equations:

$$2x = \lambda_1 + \lambda_2 \tag{2.6}$$
$$2y = -\lambda_1 + \lambda_2 \tag{2.7}$$
$$2z = \lambda_2 \tag{2.8}$$
$$x^2 + y^2 + z^2 = 11 \tag{2.9}$$
$$x + y + z = 3 \tag{2.10}$$

[1]Recall that *two* vectors are linearly dependent if and only if one is a scalar multiple of the other.

Adding (2.6) and (2.7), then using (2.8) gives

$$2x + 2y = 2\lambda_2 = 4z \implies x + y = 2z.$$

Substituting this into (2.10) gives

$$2z + z = 3 \implies z = 1.$$

Therefore

$$x + y = 2 \implies x = 2 - y.$$

Substituting into (2.9) then gives

$$\begin{gathered}(2-y)^2 + y^2 + 1^2 = 11 \\ \implies y^2 - 2y - 3 = 0 \\ \implies (y-3)(y+1) = 0.\end{gathered}$$

Thus $y = 3$ or $y = -1$, giving $x = -1$ or $x = 3$, respectively. Therefore, the extrema occur at

$$(-1, 3, 1) \quad \text{and} \quad (3, -1, 1).$$

Since $f(-1, 3, 1) = -1 - 3 = -4$ and $f(3, -1, 1) = 3 - (-1) = 4$, we see that $f|_S$ attains its minimum at $(-1, 3, 1)$ and its maximum at $(3, -1, 1)$. △

Exercises.

Exercises from [FRYb, §2.10]: Q10, Q11, Q14, Q26.

2.6 Finding global extrema

Let's now combine all the things we have learned in this chapter. So far we have developed techniques for finding *local* extrema. How do we find *global* extrema of a function f on a set S? Since every global extremum is a local extremum, the two processes are related. However, since not every local extremum is a global extremum, we still have some work to do.

The first step is to determine whether or not the global extrema even exist! For this, one of our most powerful tools is the extreme value theorem (Theorem 2.1.7). If S is compact (recall that this means it bounded and contains its boundary) and f is continuous, the extreme value theorem tells us that f does indeed attain a global maximum and minimum. Thus, we only need to find the local extrema, and then see which of them is the largest (this will be the global maximum) and which is the smallest (this will be the global minimum).

The situation is much more complicated if S is not compact. If S is not compact, then we need to look at the limiting behaviour of f as $\|\mathbf{x}\| \to \infty$ or $\mathbf{x} \to \partial S$ to decide if f is even bounded and, if so, whether or not it has a global maximum and/or minimum. Doing this is usually not so hard for functions of a single variable, but generally much harder for multivariable functions.

In this course, we will focus mainly on the case where S is compact. Typically, our set S will be a compact set of the form

$$S = \{\mathbf{x} : g(\mathbf{x}) \le k\}$$

for some C^1 function $g\colon A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$, and some scalar $k \in \mathbb{R}$. Then the boundary will often (but not always!) be given by

$$\partial S = \{\mathbf{x} : g(\mathbf{x}) = k\}.$$

We then split our search for the local extrema into two parts:

(a) We find the critical points in $S \setminus \partial S$. For this we use the first derivative test.

(b) We find the critical points in the boundary ∂S. For this we use the method of Lagrange multipliers.

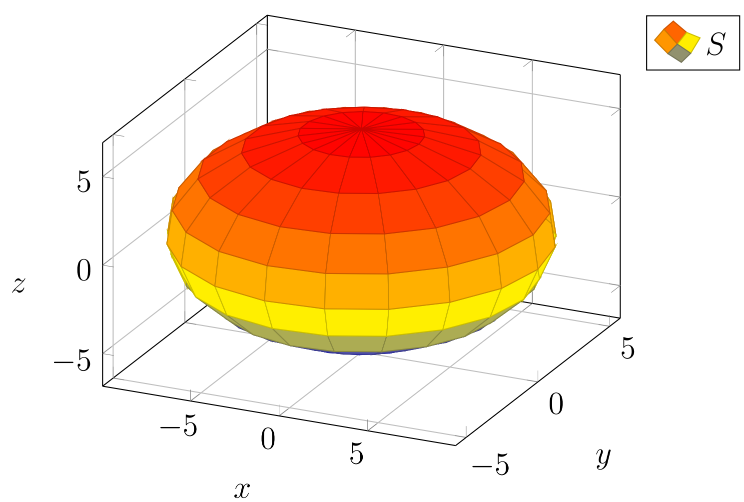
Example 2.6.1. Let's find the global extrema (if they exist) of

$$f(x, y, z) = x^2 + y^3 + z^3$$

on the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) := x^2 + 3y^2 + 3z^2 \le 100\}.$$

So S is a solid ellipsoid:



The set S is bounded and contains its boundary

$$\partial S = \{(x, y, z) : g(x, y, z) = 100\}.$$

Therefore, by the extreme value theorem (Theorem 2.1.7), f attains a global minimum and maximum on S.

First we look for critical points for f in the interior, $S \setminus \partial S$. So we solve

$$(0,0,0) = \nabla f = (2x, 3y^2, 3z^2) \implies x = y = z = 0.$$

Don't forget to check that the points you find are actually in S! Here we have

$$0^2 + 2 \cdot 0^2 + 3 \cdot 0^2 = 0 \leq 100,$$

and so $(0,0,0) \in S$.

Now we consider the boundary of S, which is the ellipsoid

$$\partial S = \{(x,y,z) \in \mathbb{R}^3 : g(x,y,z) = 100\}.$$

We have

$$\nabla g = (2x, 6y, 6z).$$

Note that $\nabla g = (0,0,0)$ would imply that $x = y = z = 0$. Since $(0,0,0) \notin \partial S$, we see that ∇g is never zero on ∂S. So ∇f and ∇g are linearly dependent if and only if $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$. Therefore the Lagrange equations become:

$$2x = 2\lambda x \tag{2.11}$$
$$3y^2 = 6\lambda y \tag{2.12}$$
$$3z^2 = 6\lambda z \tag{2.13}$$
$$x^2 + 3y^2 + 3z^2 = 100 \tag{2.14}$$

From (2.11) to (2.13), we have

$$\begin{aligned} (\lambda - 1)x = 0 &\implies x = 0 \text{ or } \lambda = 1, \\ (y - 2\lambda)y = 0 &\implies y = 0 \text{ or } y = 2\lambda, \\ (z - 2\lambda)z = 0 &\implies z = 0 \text{ or } z = 2\lambda. \end{aligned}$$

Considering all the above possibilities, together with the constraint (2.14), we have the following critical points (including the one we found in the interior of S), together with the corresponding value of the function f:

Critical point	$f(x,y,z)$
$(0,0,0)$	0
$(0,0,\pm 10/\sqrt{3})$	$\pm 1000/3\sqrt{3}$
$(0,\pm 10\sqrt{3},0)$	$\pm 1000/3\sqrt{3}$
$\pm(0,5/\sqrt{6},5/\sqrt{6})$	$125/3\sqrt{6}$
$(\pm 10,0,0)$	100
$(\pm 2\sqrt{19},2,2)$	92
$(\pm 2\sqrt{22},0,2)$	96
$(\pm 2\sqrt{22},2,0)$	96

Note that $1000/3\sqrt{3} \approx 192.45$ and $125/3\sqrt{6} \approx 17.01$. Thus, f attains its maximum value of $1000/3\sqrt{3}$ at the points $(0,0,10/\sqrt{3})$ and $(0,10/\sqrt{3},0)$ and it attains its minimum value of $-1000/3\sqrt{3}$ at the points $(0,0,-10/\sqrt{3})$ and $(0,-10/\sqrt{3},0)$. △

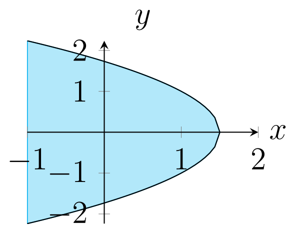
Example 2.6.2. Let's find the global extrema, if they exist, of

$$f(x, y) = x^2 + 2y$$

on the set

$$S = \{(x, y) \in \mathbb{R}^2 : g(x, y) := 2x + y^2 \leq 3\}.$$

The set S looks as follows:



Its boundary is

$$\partial S = \{(x, y) : g(x, y) = 3\}.$$

Note that S is *not* bounded, and so we *cannot* apply the extreme value theorem (Theorem 2.1.7) to conclude that f has a global maximum or minimum on the set S.

Nevertheless, we begin by finding the critical points in the interior of S. Since

$$\nabla f = (2x, 2) \neq (0, 0),$$

there are no such critical points.

Next, we use the method of Lagrange multipliers to find the critical points on the boundary. Since $\nabla g = (2, 2y)$ is never zero, the Lagrange equations become:

$$2x = 2\lambda \tag{2.15}$$

$$2 = 2\lambda y \tag{2.16}$$

$$2x + y^2 = 3 \tag{2.17}$$

The equation (2.15) gives that $x = \lambda$, while (2.16) implies that $y = \frac{1}{\lambda}$. (Note that $\lambda = 0$ contradicts (2.16), so we can divide by λ.) Substituting into (2.17) gives

$$2\lambda + \frac{1}{\lambda^2} = 3 \implies \frac{2}{y} + y^2 = 3.$$

(Note that $y = 0$ would contradict (2.16), and so we have $y \neq 0$.) Multiplying by y, we have

$$y^3 - 3y + 2 = 0.$$

Since $y = 1$ is clearly a solution, we can use polynomial division to find the other factors:

$$y^3 - 3y + 2 = (y - 1)(y^2 + y - 2) = (y - 1)^2(y + 2).$$

Thus, $y = 1$ or $y = -2$. When $y = 1$, we have $x = \lambda = \frac{1}{y} = 1$. When $y = -2$, we have $x = -\frac{1}{2}$. Thus, the critical points on ∂S are

$$(1, 1) \quad \text{and} \quad \left(-\tfrac{1}{2}, -2\right).$$

Now,

$$f(1,1) = 1^2 + 2\cdot 1 = 3 \quad \text{and} \quad f\left(-\tfrac{1}{2}, -2\right) = \left(-\tfrac{1}{2}\right)^2 + 2(-2) = -\tfrac{15}{4}.$$

Are the points we found global extrema? Note that $(x,0) \in S$ for arbitrarily large negative x. Since $f(x,0) = x^2$, this means that f takes arbitrarily large values on S. So f has no global maximum. Since there is no other critical point on S, the function f *does* attain a minimum value of $-\frac{15}{4}$ at the point $\left(-\frac{1}{2}, -2\right)$. (We use here the fact that $f(x,y) \to \infty$ as $\|(x,y)\| \to \infty$.) $\triangle$

Exercises.

2.6.1. Find the absolute maximum and minimum values of $f(x,y) = x^2 + 3xy + y^2 + 7$ on $D = \{(x,y) : x^2 + y^2 \leq 1\}$.

2.6.2. Find and classify the extreme values (if any) of the following functions defined on $\mathbb{R}^2$:

(a) $x^2 - y^5$

(b) $(x+1)^2 + (x-y)^2$

(c) $x^2 + xy^2 + y^4$

2.6.3. Find the maximum and minimum values of $f(x,y) = xy - y + x - 1$ subject to the condition $x^2 + y^2 \leq 2$.

Chapter 3

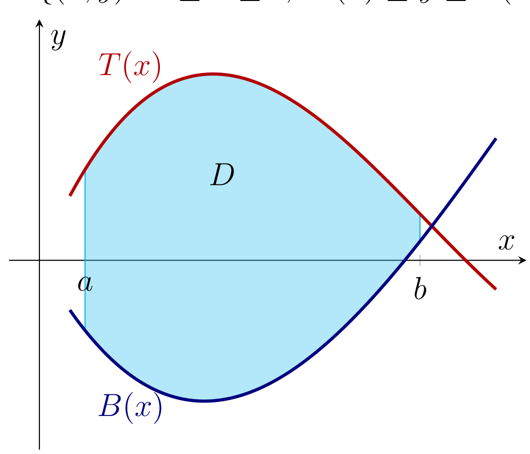
Double and triple integrals

In previous calculus courses, you learned about single variable integrals, such as $\int_a^b f(x)\,dx$. In this chapter we will explore integration in two and three dimensions. We begin in Sections 3.1 and 3.2 with an informal treatment, discussing how one computes double integrals in practice (via iterated integrals). We then give a more precise definition of the double integral in Sections 3.3 to 3.5. Finally, in Section 3.7, we discuss integration in three dimensions. A good reference of the material in this chapter is [FRYc, §3.1–3.4].

3.1 Vertical slices

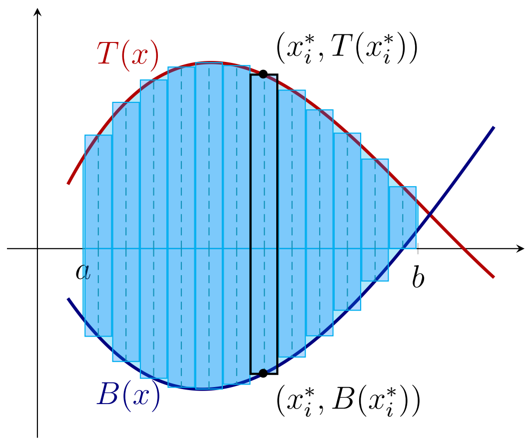
Suppose we want to compute the mass of a two-dimensional object occupying the region D in the xy-plane. We first need to know the density of the object, which may depend on position. Let $f(x, y)$ denote the density at the point (x, y). For simplicity, let's assume that D is the region between the bottom curve $y = B(x)$ and the top curve $y = T(x)$, with x running from a to b. Thus,

$$D = \{(x, y) : a \leq x \leq b,\ B(x) \leq y \leq T(x)\}.$$



Now, if we wanted to compute the *area* of D (instead of its mass), we learned in single variable calculus that we should approximate the region by rectangles, which we call *slices*. We do this by splitting the interval $[a, b]$ into n subintervals and choosing a point x_i^* in the

i-th interval. Then, for each interval, we draw a rectangle over that interval, with top edge at height $T(x_i^*)$ and bottom edge at height $B(x_i^*)$.



The sum of the areas of the slices (called a *Riemann sum*) is

$$\sum_{i=1}^{n} (T(x_i^*) - B(x_i^*))\Delta x \quad \text{where } \Delta x = \frac{b-a}{n}.$$

We then take the limit as $n \to \infty$. If this limit exists, it is the area between the two curves:

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} (T(x_i^*) - B(x_i^*))\Delta x = \int_a^b (T(x) - B(x))\, dx\,.$$

We need to modify this procedure in order to calculate the mass of D, instead of its area. To do this, we need to replace our approximation $(T(x_i^*) - B(x_i^*))\Delta x$ of the area of the i-th slice of D by an approximation of the mass of this slice. To do this, we subdivide the slice into m rectangles of height

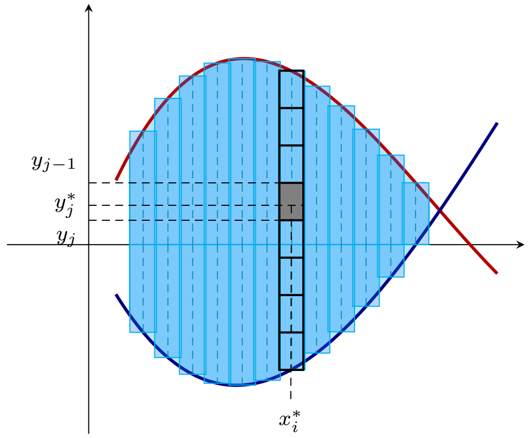
$$\Delta y = \frac{T(x_i^*) - B(x_i^*)}{m}.$$

Let

$$y_j = B(x_i^*) + j\Delta y$$

be the y-coordinate of the top of the j-th rectangle. For each $j = 1, \ldots, m$, we choose a number y_j^* between y_{j-1} and y_j. We then approximate the density on rectangle j in slice i

by the constant $f(x_i^*, y_j^*)$.



The mass of rectangle j in slice i is approximately

$$f(x_i^*, y_i^*)\Delta x \Delta y.$$

Taking the limit as $m \to \infty$ (that is, the limit as the height of the rectangles goes to zero), the Riemann sum becomes an integral:

$$\text{Mass of slice } i \approx \Delta x \int_{B(x_i^*)}^{T(x_i^*)} f(x_i^*, y)\, dy = F(x_i^*)\Delta x,$$

where

$$F(x) = \int_{B(x)}^{T(x)} f(x, y)\, dy\,.$$

Notice that we started with a function $f(x, y)$, and we have “integrated out” the variable y, leaving a function $F(x)$ of one variable.

Now we take the limit at $n \to \infty$ (that is, the limit as the width of the slices goes to zero), giving

$$\text{Mass} = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \int_{B(x_i^*)}^{T(x_i^*)} f(x_i^*, y)\, dy = \lim_{n\to\infty} \sum_{i=1} F(x_i^*)\Delta x.$$

Now, the sum

$$\sum_{i=1} F(x_i^*)\Delta x$$

is a Riemann sum approximation to the integral $\int_a^b F(x)\, dx$. Thus,

$$\text{Mass} = \int_a^b F(x)\, dx = \int_a^b \left(\int_{B(x)}^{T(x)} f(x, y)\, dy \right) dx\,.$$

Notation 3.1.1 (Double integral).

$$\iint\limits_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{B(x)}^{T(x)} f(x,y)\,dy \right) dx$$
$$= \int_a^b \int_{B(x)}^{T(x)} f(x,y)\,dy\,dx = \int_a^b dx \int_{B(x)}^{T(x)} f(x,y)\,dy\,.$$

The last three integrals are called *iterated integrals*. Note the placement of dx and dy in these iterated integrals. The integral signs (and their limits of integration) match up with the dx or dy in the same way that you match parentheses in a mathematical expression.

Example 3.1.2. Let D be the region above the curve $x^2 = 9y$ and to the right of the curve $3y^2 = x$. Let's compute

$$\iint\limits_D xy\,dx\,dy\,.$$

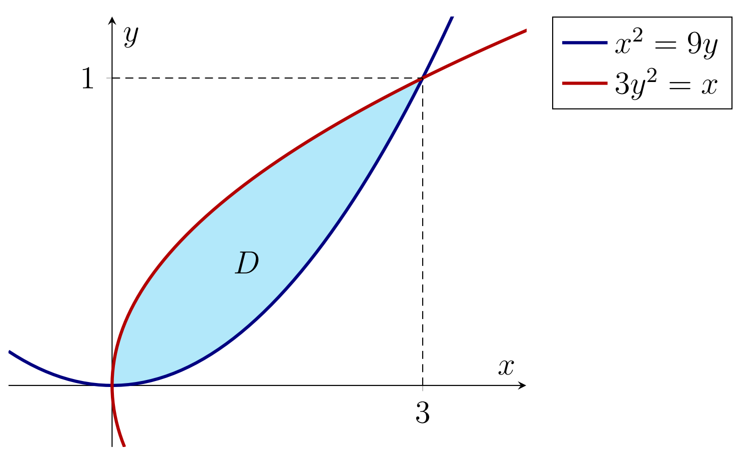
The curves intersect at points (x,y) that satisfy both

$$x^2 = 9y \qquad \text{and} \qquad 3y^2 = x.$$

Thus, we have

$$y = \frac{x^2}{9} = \frac{(3y^2)^2}{9} = y^4 \implies y^4 - y = 0 \implies y(y^3-1) = 0 \implies y = 0 \text{ or } y = 1.$$

Therefore, the curves intersect at $(0,0)$ and $(3,1)$. So the region D looks as follows:



For $0 \le x \le 3$, the bottom boundary of the region D is given by $y = \frac{x^2}{9}$ and the top boundary is given by $y = \sqrt{\frac{x}{3}}$. Thus, we have

$$\iint\limits_D xy\,dx\,dy = \int_0^3 \left(\int_{x^2/9}^{\sqrt{x/3}} xy\,dy \right) dx$$

$$
\begin{aligned}
&= \int_0^3 \left(\left[\frac{1}{2} x y^2 \right]_{y=x^2/9}^{\sqrt{x/3}} \right) dx \\
&= \int_0^3 \left(\frac{x^2}{6} - \frac{x^5}{162} \right) dx \\
&= \left[\frac{x^3}{18} - \frac{x^6}{6 \cdot 162} \right]_0^3 \\
&= \frac{3}{2} - \frac{3}{4} = \frac{3}{4}.
\end{aligned}
$$

Note that we first integrated with respect to y, treating x as a constant. Then we integrated with respect to x. △

Example 3.1.3. If $f(x, y) = 1$ for all $(x, y) \in D$, then we have

$$\iint_D f(x, y)\, dx\, dy = \int_a^b dx \int_{B(x)}^{T(x)} dy = \int_a^b [y]_{B(x)}^{T(x)}\, dx = \int_a^b (T(x) - B(x))\, dx\,.$$

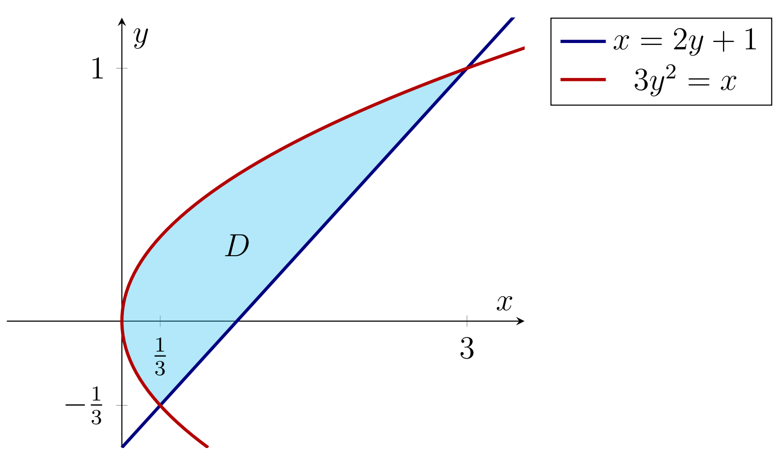
This is precisely the formula for the area between two curves that you learned in single-variable calculus! △

Exercises.

Exercises are deferred until Section 3.6.

3.2 Horizontal slices

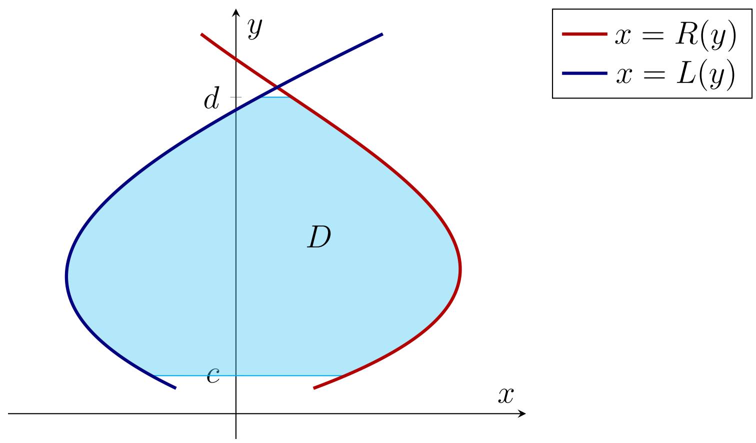
In Section 3.1, to integrate over a region D, we approximated the region by vertical slices. As you may expect, we can also approximate the region by *horizontal* slices. It is quite useful to have the flexibility to use either method. For instance, consider the following region:



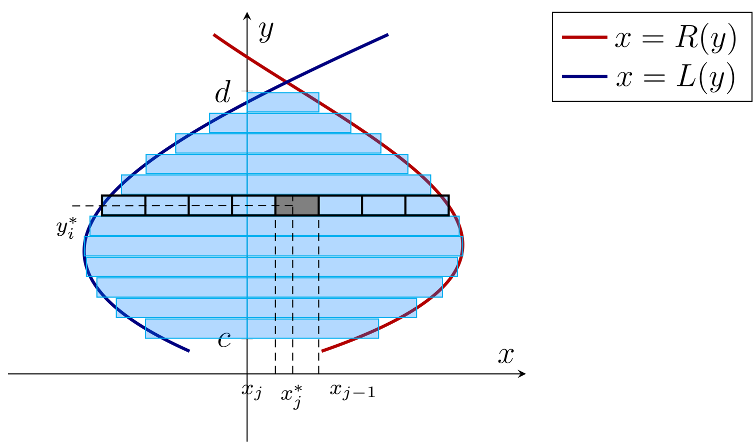
Using the vertical slice method, we would have to split the region D into two pieces. For $0 \le x \le \frac{1}{3}$, the region is bounded above by the curve $y = \sqrt{\frac{x}{3}}$ and below by the curve $y = -\sqrt{\frac{x}{3}}$. Then, for $\frac{1}{3} \le x \le 3$, the region is bounded above by $y = \sqrt{\frac{x}{3}}$ and below by $y = \frac{x-1}{2}$. On the other hand, the region D is more simply described as the region to the right of the curve $x = 2y + 1$ and to the left of the curve $3y^2 = x$.

Let's consider the general setup. Suppose we want to integrate the function $f(x, y)$ over the region

$$D = \{(x, y) : c \le y \le d,\ L(y) \le x \le R(y)\}.$$



To integrate using horizontal slices, we follow the procedure of Section 3.1, but we draw our slices horizontally and reverse the roles of x and y everywhere.



Interchanging x and y in our discussion from Section 3.1 then tells us that, in order to integrate the function $f(x, y)$ over the region D, we compute the double integral

$$\int_c^d \left(\int_{L(y)}^{R(y)} f(x, y)\, dx \right) dy\,.$$

The following notation should be compared to Notation 3.1.1.

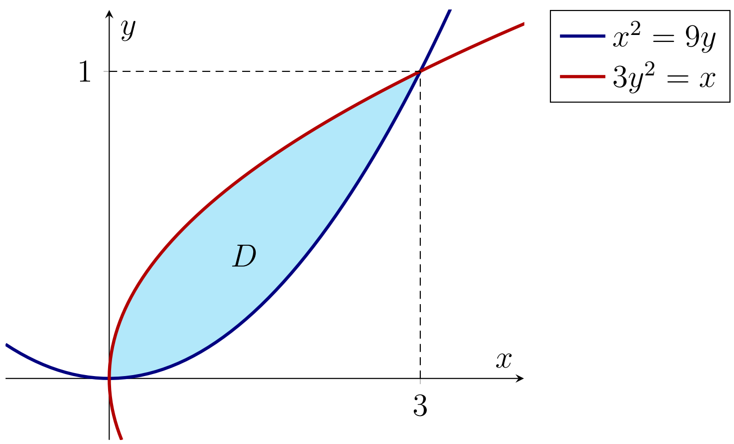
Notation 3.2.1 (Double integral).

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \left(\int_{L(y)}^{R(y)} f(x,y)\,dx \right) dy$$
$$= \int_c^d \int_{L(y)}^{R(y)} f(x,y)\,dx\,dy = \int_c^d dy \int_{L(y)}^{R(y)} f(x,y)\,dx\,.$$

Example 3.2.2. Let's compute the integral from Example 3.1.2, but using horizontal slices instead of vertical slices. So we want to compute

$$\iint_D xy\,dx\,dy$$

where D is the following region:



In order to use the method of horizontal slices, we describe the region as

$$D = \{(x,y) : 0 \le y \le 1,\ 3y^2 \le x \le 3\sqrt{y}\}.$$

Thus we have

$$\begin{aligned} \iint_D xy\,dx\,dy &= \int_0^1 \left(\int_{3y^2}^{3\sqrt{y}} xy\,dx \right) dy \\ &= \int_0^1 \left(\left[\frac{1}{2}x^2 y \right]_{x=3y^2}^{3\sqrt{y}} \right) dy \\ &= \int_0^1 \left(\frac{9y^2}{2} - \frac{9y^5}{2} \right) dy \\ &= \left[\frac{3y^3}{2} - \frac{3y^6}{4} \right]_0^1 \\ &= \frac{3}{2} - \frac{3}{4} = \frac{3}{4}. \end{aligned}$$

Note that we got the same answer as we did in Example 3.1.2. We will see *why* this happened in Theorem 3.4.1. △

Exercises.

Exercises are deferred until Section 3.6.

3.3 Double integrals over a rectangle

In Sections 3.1 and 3.2 we developed two ways to compute a double integral over a region in the plane. In Example 3.1.2 and Example 3.2.2 we computed the same integral in both ways and got the same answer. Of course, this is what we want to happen. But we haven't yet explained why this should be the case. In this section, we give an alternative definition of the double integral, and then give some conditions that ensure that this definition matches the calculation via vertical and horizontal slices.

Consider a closed rectangle

$$R = [a, b] \times [c, d] \subseteq \mathbb{R}^2.$$

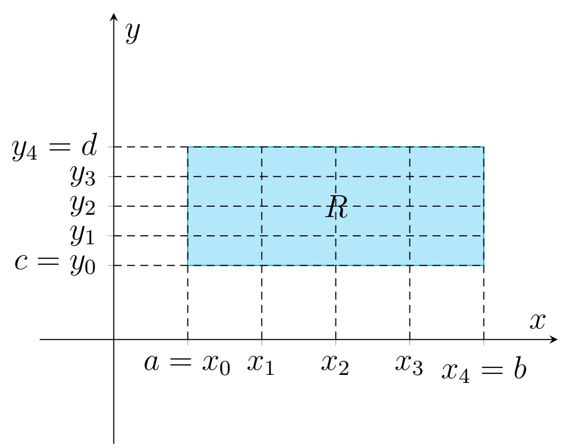
A *regular partition* of R of order n is the collection of points $\{x_j\}_{j=0}^n$ and $\{y_j\}_{j=0}^n$ satisfying

$$a = x_0 < x_1 < \cdots < x_n = b, \qquad c = y_0 < y_1 < \cdots < y_n = d,$$

where

$$x_{j+1} - x_j = \frac{b-a}{n}, \qquad y_{k+1} - y_k = \frac{d-c}{n}$$

for $0 \le j, k \le n-1$. Using these equally spaced points, we subdivide the rectangle R:



Let

$$R_{jk} = [x_j, x_{j+1}] \times [y_k, y_{k+1}]$$

and let $\mathbf{c}_{jk}$ be *any* point in R_{jk}. Suppose $f \colon R \to \mathbb{R}$ is bounded. For example, since R is compact, it follows from Lemma 2.1.9 that f is bounded if f is continuous. Form the sum

$$S_n = \sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk}) \Delta x \Delta y = \sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk}) \Delta A, \tag{3.1}$$

where

$$\Delta x = \frac{b-a}{n}, \qquad \Delta y = \frac{d-c}{n}, \qquad \Delta A = \Delta x \Delta y.$$

Thus ΔA is the area of each of the subrectangles into which we have divided R. The sum (3.1) is called a *Riemann sum* for f.

We can now give the precise definition of the double integral.

Definition 3.3.1 (Double integral)**.** If the sequence $(S_n)_{n=1}^{\infty}$ converges to a limit S as $n \to \infty$ and if the limit S is the same for any choice of points $\mathbf{c}_{jk}$ in the rectangle R_{jk}, then we say that f is *integrable* over R, and we write

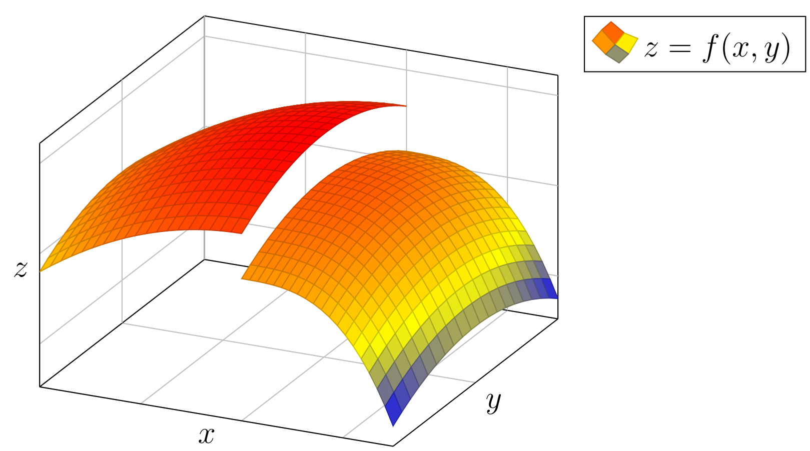
$$\iint\limits_R f(x,y)\, dA, \qquad \iint\limits_R f(x,y)\, dx\, dy, \quad \text{or} \qquad \iint\limits_R f\, dx\, dy\,.$$

for the limit S.

Note that some of the notation in Definition 3.3.1 is the same as the Notation 3.1.1 and 3.2.1. We should justify why we can use the same notation. It is also very useful to have a class of functions which we know are integrable.

Theorem 3.3.2. *Any continuous function defined on a closed rectangle is integrable.*

In fact, a more general result is true. Consider a function $f \colon R \to \mathbb{R}$ whose graph looks something like the following:



This function is continuous every except along a continuous curve (in the domain R). It turns out that this type of function is also integrable.

Theorem 3.3.3. *Suppose R is a closed rectangle and $f \colon R \to \mathbb{R}$ is bounded. Furthermore, suppose that the set of points where f is discontinuous lies on a finite union of graphs of continuous functions. Then f is integrable over R.*

Proof. We will not prove this theorem in this course. The basic idea behind the proof is the same as that for single-variable functions, where a function with only finitely many discontinuities is integrable. In the limit defining the integral, the terms involving the discontinuities become zero. $\square$

Theorem 3.3.3 will be key in handling integrals over more general regions, and relating Definition 3.3.1 to the vertical and horizontal slice methods of Sections 3.1 and 3.2. Before we do this, we state some properties of integrals that follows from Definition 3.3.1 and properties of limits.

Proposition 3.3.4 (Arithmetic of integrals)**.** *Suppose f and g are integrable functions on a rectangle R, and let $c \in \mathbb{R}$. Then $f+g$ and cf are integrable, and we have the following.*

(a) Linearity*:*

$$\iint\limits_R (f(x,y) + g(x,y))\,dA = \iint\limits_R f(x,y)\,dA + \iint\limits_R g(x,y)\,dA\,.$$

(b) Homogeneity*:*

$$\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA\,.$$

(c) Monotonicity*: If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then*

$$\iint\limits_R f(x,y)\,dA \geq \iint\limits_R g(x,y)\,dA\,.$$

Corollary 3.3.5. *If f is integrable on a rectangle R, then so is $|f|$, and we have*

$$\left|\iint\limits_R f\,dA\right| \leq \iint\limits_R |f|\,dA\,. \tag{3.2}$$

Proof. Since

$$-|f(x,y)| \leq f(x,y) \leq |f(x,y)| \quad \text{for all } (x,y) \in R,$$

we can use monotonicity and homogeneity of integration (with $c = -1$) to conclude that

$$-\iint\limits_R |f|\,dA \leq \iint\limits_R f\,dA \leq \iint\limits_R |f|\,dA,$$

which is equivalent to (3.2). □

Exercises.

Exercises are deferred until Section 3.6.

3.4 Fubini's theorem

The following theorem states that, for continuous functions, one can compute the integral over a rectangle by the vertical slice method from Section 3.1 or the horizontal slice method from Section 3.2. In particular, the vertical and horizontal slice methods give the same answer, explaining the phenomenon we say in Example 3.1.2 and Example 3.2.2.

Theorem 3.4.1 (Fubini's theorem, continuous case)**.** *Suppose* $R = [a, b] \times [c, d]$ *and* $f \colon R \to \mathbb{R}$ *is continuous. Then*

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy\,.$$

Proof. For a proof of this theorem, see [MT11, Th. 5.2.3]. □

Example 3.4.2. Let $R = [0, 3] \times [e, 2e]$, and let

$$f \colon R \to \mathbb{R}, \quad f(x,y) = \frac{x}{y}.$$

By Fubini's theorem (Theorem 3.4.1), we have

$$\begin{aligned}
\iint\limits_R f\,dA &= \int_e^{2e} \int_0^3 f(x,y)\,dx\,dy \\
&= \int_e^{2e} \int_0^3 \frac{x}{y}\,dx\,dy \\
&= \int_e^{2e} \left[\frac{x^2}{2y}\right]_{x=0}^3 dy \\
&= \int_e^{2e} \frac{9}{2y}\,dy \\
&= \int_e^{2e} \frac{9}{2y}\,dy \\
&= \left[\frac{9}{2}\ln y\right]_{y=e}^{2e} \\
&= \frac{9}{2}\big(\ln(2e) - \ln e\big) \\
&= \frac{9}{2}\big(\ln 2 + \ln e - \ln e\big) \\
&= \frac{9}{2}\ln 2.
\end{aligned}$$

Alternatively, we could first integrate with respect to y, giving

$$\iint\limits_R f\,dA = \int_0^3 \int_e^{2e} \frac{x}{y}\,dy\,dx$$

$$\begin{aligned}
&= \int_0^3 [x \ln y]_{y=e}^{2e} \, dx \\
&= \int_0^3 x \ln 2 \, dx \\
&= \left[\frac{1}{2} x^2 \ln 2 \right]_{x=0}^3 \\
&= \frac{9}{2} \ln 2.
\end{aligned}$$

As expected, we obtain the same answer. △

As we saw in Theorem 3.3.3, a function that is discontinuous only on a finite union of graphs of continuous functions is still integrable. The following theorem generalizes Theorem 3.4.1 to this setting.

Theorem 3.4.3 (Fubini's theorem, general version)**.** *Suppose $R = [a,b] \times [c,d]$ and $f \colon R \to \mathbb{R}$ is a bounded function whose discontinuities lie on a finite union of graphs of continuous functions.*

(a) *If the integral $\int_c^d f(x,y)\,dy$ exists for all $x \in [a,b]$, then*

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

exists and

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \iint\limits_R f(x,y)\,dA\,.$$

(b) *If the integral $\int_a^b f(x,y)\,dx$ exists for all $y \in [c,d]$, then*

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \iint\limits_R f(x,y)\,dA\,.$$

Therefore, if all the above conditions hold, we have

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy = \iint\limits_R f(x,y)\,dA\,.$$

We now give an example of a "problematic function".

Example 3.4.4. Let $R = [0,1] \times [0,1]$ and

$$f \colon R \to \mathbb{R}, \quad f(x,y) = \begin{cases} 1 & \text{if } x, y \text{ are both rational numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the integral $\iint\limits_R f\,dA$ does *not* exist. This is because, if we choose the points $\mathbf{c}_{jk}$ in Definition 3.3.1 to have rational coordinates, the Riemann sums are all equal to 1, whereas if we choose them to have irrational coordinates, the Riemann sums are all equal to 0. △

Remark 3.4.5. If you take more advanced courses in analysis, you will learn about the Lebesgue integral which *does* exist for the function f of Example 3.4.4, and is equal to 0, essentially because there are way more irrational numbers than rational numbers.

Exercises.

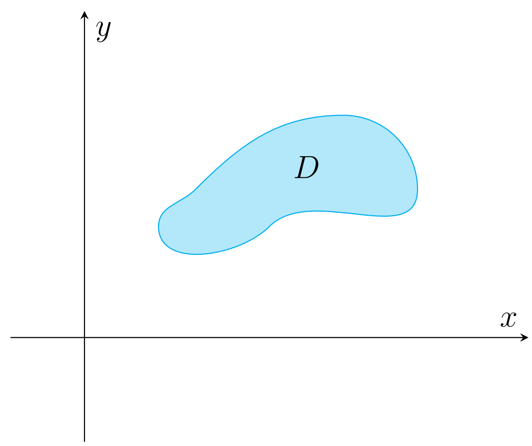
Exercises from [FRYb, §3.1]: Q1, Q2, Q3a

Further exercises are deferred until Section 3.6.

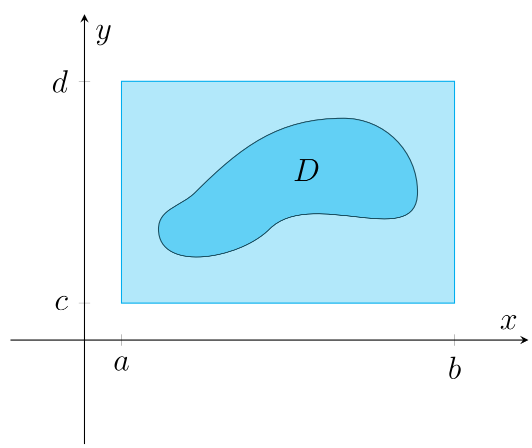
3.5 Double integrals over more general regions

Fubini's theorem (Theorem 3.4.3) told us that the integral over a rectangle (in the sense of Definition 3.3.1) is equal to the iterated integrals corresponding to the vertical and horizontal slice methods. However, it is stated only for rectangular regions. We'd like to be able to work with more general regions in the plane. Fortunately, there is an easy "trick" allowing us to do this.

Suppose D is a bounded region in $\mathbb{R}^2$, and we want to integrate a function $f\colon D \to \mathbb{R}$ over D.



Since D is bounded, we can enclose it in a rectangle $R = [a, b] \times [c, d]$.



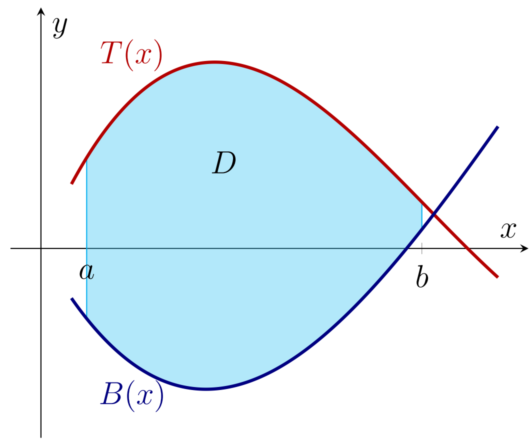
Our "trick" is to extend the function f to R by defining it to be zero outside of D. Let us make this more precise.

We say a set $D \subseteq \mathbb{R}^2$ is *y-simple* if there are two continuous functions

$$B, T\colon [a, b] \to \mathbb{R}$$

such that

$$D = \{(x, y) : a \leq x \leq b,\ B(x) \leq y \leq T(x)\}.$$



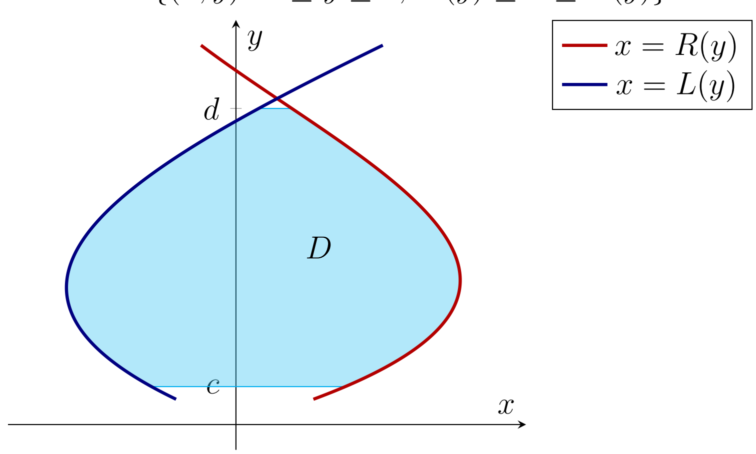
We considered these types of regions in Section 3.1.

We say a set $D \subseteq \mathbb{R}^2$ is *x-simple* if there are continuous functions

$$L, R\colon [c, d] \to \mathbb{R}$$

such that

$$D = \{(x, y) : c \leq y \leq d,\ L(y) \leq x \leq R(y)\}.$$



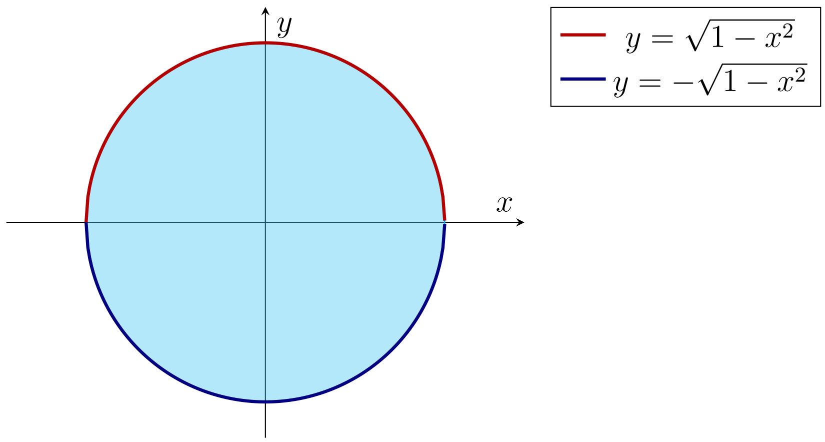
These are the types of regions we considered in Section 3.2.

We say a set $D \subseteq \mathbb{R}^2$ is *simple* if it is both x-simple and y-simple. For instance the unit disk

$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$

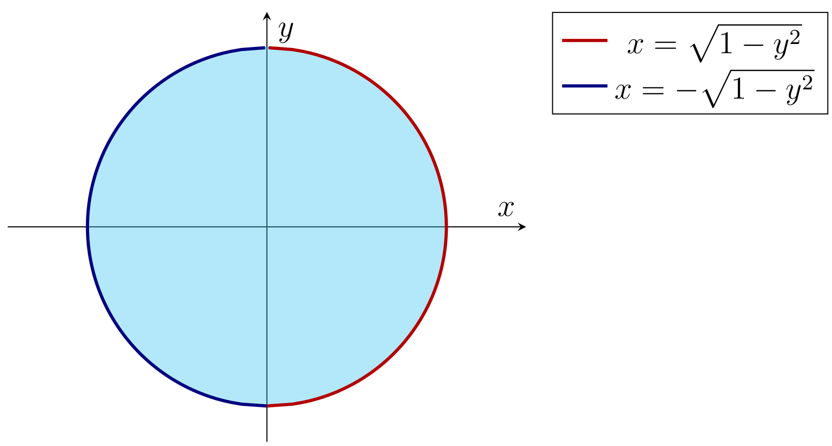
since we have

$$D = \{(x, y) : -1 \leq x \leq 1,\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$$



and

$$D = \{(x, y) : -1 \leq y \leq 1,\ -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\}$$



A set $D \subseteq \mathbb{R}^2$ is an *elementary region* if it is either x-simple or y-simple (or both).

We can now give the precise definition of the integral over a more general region in the plane.

Definition 3.5.1 (Double integral over an elementary region)**.** Suppose D is an elementary region in the plane and R is a rectangle containing D. Furthermore, suppose $f: D \to \mathbb{R}$ is continuous. Define

$$f^*: R \to \mathbb{R}, \quad f^*(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Then f^* is integrable over R by Theorem 3.3.3. We define the *integral of f over D* to be

$$\iint\limits_D f(x,y)\, dA = \iint\limits_R f^*(x,y)\, dA\,.$$

Definition 3.5.1 is related to the vertical and horizontal slice methods of Sections 3.1 and 3.2 by the following result.

Theorem 3.5.2 (Reduction to iterated integrals). *Suppose $f\colon D \to \mathbb{R}$ is a continuous function.*

(a) *If*

$$D = \{(x,y) : a \le x \le b,\ B(x) \le y \le T(x)\}$$

is a y-simple region, then

$$\iint\limits_D f(x,y)\,dA = \int_a^b \int_{B(x)}^{T(x)} f(x,y)\,dy\,dx\,.$$

(b) *If*

$$D = \{(x,y) : c \le y \le d,\ L(y) \le x \le R(y)\}$$

is an x-simple region, then

$$\iint\limits_D f(x,y)\,dA = \int_c^d \int_{L(y)}^{R(y)} f(x,y)\,dx\,dy\,.$$

We can now add one more important property of integrals to those listed in Proposition 3.3.4. Recall that two sets X and Y are *disjoint* if $X \cap Y = \varnothing$.

Proposition 3.5.3 (Additivity of the integral). *If $D_1, D_2 \subseteq \mathbb{R}^2$ are disjoint elementary regions and $f\colon D_1 \cup D_2 \to \mathbb{R}$ is a continuous function, then*

$$\iint\limits_{D_1 \cup D_2} f(x,y)\,dA = \iint\limits_{D_1} f(x,y)\,dA + \iint\limits_{D_2} f(x,y)\,dA\,.$$

Exercises.

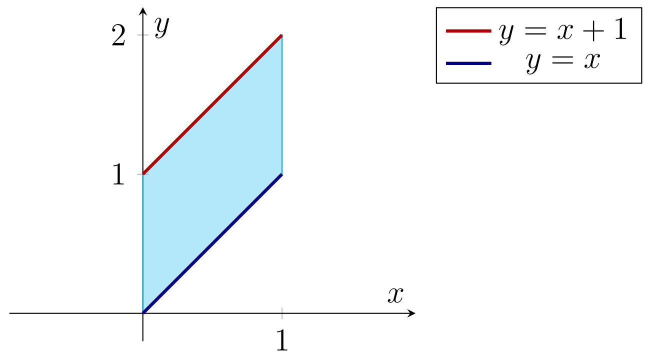
Exercises from [FRYb, §3.1]: Q3

Further exercises are deferred until Section 3.6.

3.6 Examples and applications of double integrals

Example 3.6.1. Let

$$D = \{(x,y) : 0 \le x \le 1,\ x \le y \le x+1\}$$



and

$$f\colon D \to \mathbb{R}, \quad f(x,y) = 2y - (x+1)^2.$$

Then

$$\begin{aligned}\iint\limits_D f\,dA &= \int_0^1 \int_x^{x+1} \big(2y - (x+1)^2\big)\,dy\,dx \\ &= \int_0^1 \big[y^2 - (x+1)^2 y\big]_{y=x}^{x+1}\,dx \\ &= \int_0^1 \big((x+1)^2 - (x+1)^3 - x^2 + x(x+1)^2\big)\,dx \\ &= \int_0^1 \big((x+1)^2(1-x-1+x) - x^2\big)\,dx \\ &= -\int_0^1 x^2\,dx \\ &= -\left[\frac{x^3}{3}\right]_0^1 \\ &= -\frac{1}{3}.\end{aligned}$$

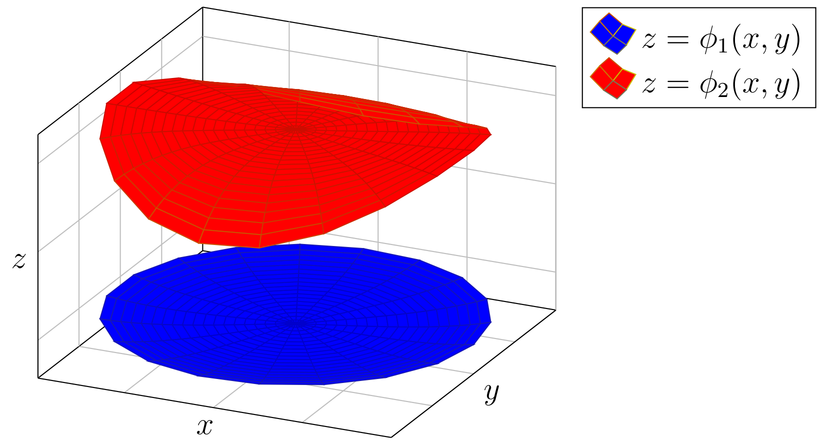
Note that D is also an x-simple region. So we could compute the integral by first integrating with respect to x. But this would involve splitting the region into two pieces. △

One important use of the double integral is computing volumes. You learned in single-variable calculus how to compute the volume of certain types of solids (e.g. solids of revolution). Using the double integral, we can compute more general volumes. In particular, suppose $D \subseteq \mathbb{R}^2$ is an elementary region, and suppose we have two functions

$$\phi_1, \phi_2\colon D \to \mathbb{R}, \quad \phi_1(x,y) \le \phi_2(x,y) \text{ for all } (x,y) \in D.$$

Consider the solid

$$S = \{(x,y,z) : (x,y) \in D,\ \phi_1(x,y) \le z \le \phi_2(x,y)\}$$

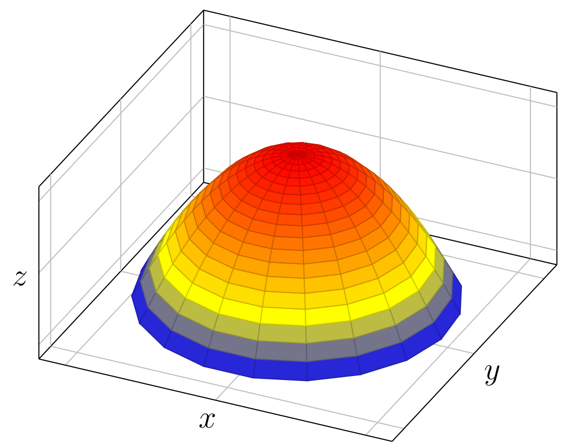


where we have drawn the graphs of the functions ϕ_1, ϕ_2 giving the top and bottom of the solid. The volume of S is given by

$$\operatorname{Vol} S := \iint_D \big(\phi_2(x, y) - \phi_1(x, y)\big)\, dA\,. \tag{3.3}$$

Example 3.6.2. Let's compute the volume of the solid

$$S = \{(x, y, z) : 0 \le z \le 1 - x^2 - y^2\}.$$



First we note that

$$S = \{(x, y, z) : (x, y) \in D,\ 0 \le z \le 1 - x^2 - y^2\},$$

where

$$D = \{(x, y) : x^2 + y^2 \le 1\}$$

is the unit disk. The region D is simple, so we can choose to describe it as an x-simple region or as a y-simple region. Let's describe it as an x-simple region:

$$D = \{(x, y) : -1 \le y \le 1,\ -\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2}\}.$$

Then we have

$$
\begin{aligned}
\operatorname{Vol} S &= \iint_D (1 - x^2 - y^2)\, dA \\
&= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (1 - x^2 - y^2)\, dx\, dy \\
&= \int_{-1}^{1} \left[(1 - y^2)x - \frac{x^3}{3} \right]_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy \\
&= 2 \int_{-1}^{1} \left((1 - y^2)\sqrt{1 - y^2} - \frac{1}{3}(1 - y^2)^{3/2} \right) dy \\
&= \frac{4}{3} \int_{-1}^{1} (1 - y^2)^{3/2}\, dy\,.
\end{aligned}
$$

We now make the substitution

$$
y = \sin t, \quad t \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2} \right],
$$

so that

$$
dy = \cos(t)\, dt \quad \text{and} \quad \sqrt{1 - y^2} = \sqrt{1 - \sin^2(t)} = \cos t.
$$

Thus

$$
\operatorname{Vol} S = \frac{4}{3} \int_{-\pi/2}^{\pi/2} \cos^4(t)\, dt\,.
$$

Now, recall the double-angle identity

$$
\cos(2t) = 2\cos^2(t) - 1 \implies \cos^2(t) = \frac{\cos(2t) + 1}{2}.
$$

So we have

$$
\begin{aligned}
\cos^4(t) = \left(\frac{\cos(2t) + 1}{2} \right)^2 &= \frac{\cos^2(2t) + 2\cos(2t) + 1}{4} \\
&= \frac{\cos(4t) + 1}{8} + \frac{\cos(2t)}{2} + \frac{1}{4} = \frac{\cos(4t)}{8} + \frac{\cos(2t)}{2} + \frac{3}{8}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\operatorname{Vol} S &= \frac{4}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{\cos(4t)}{8} + \frac{\cos(2t)}{2} + \frac{3}{8} \right) dt \\
&= \frac{4}{3} \left[\frac{\sin(4t)}{4 \cdot 8} + \frac{\sin(2t)}{2 \cdot 2} + \frac{3t}{8} \right]_{t=-\pi/2}^{\pi/2} \\
&= \frac{4}{3} \cdot \frac{3}{8} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\pi}{2}. \qquad \triangle
\end{aligned}
$$

It is sometimes useful to change the order of integration in an iterated integral. In some cases, such a change makes the integral much easier to compute.

Example 3.6.3 ([MT11, Example 5.4.2]). Let's evaluate the integral

$$\int_1^2 \int_0^{\ln x} (x-1)\sqrt{1+e^{2y}}\, dy\, dx\,. \tag{3.4}$$

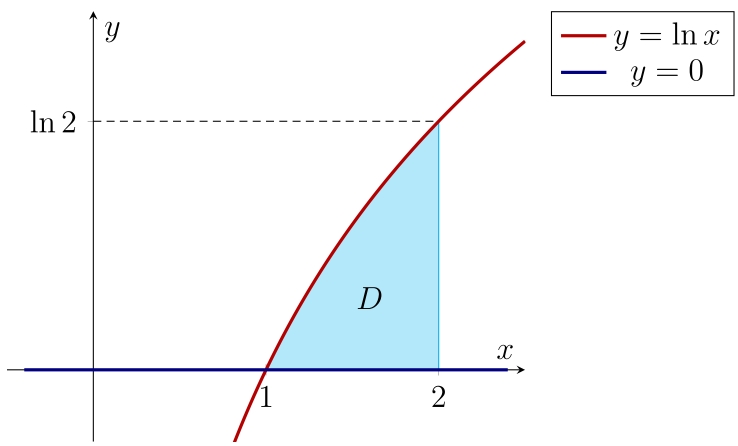
This iterated integral is quite hard to compute as it is. However, it will become easier once we change the order of integration.

We first note that the above integral is equal to

$$\iint_D (x-1)\sqrt{1+e^{2y}}\, dA,$$

where

$$D = \{(x,y) : 1 \le x \le 2,\ 0 \le y \le \ln x\}.$$



The region D is simple, and is also given by

$$D = \{(x,y) : 0 \le y \le \ln 2,\ e^y \le x \le 2\}.$$

Thus, the integral (3.4) is equal to

$$\begin{aligned}
\int_0^{\ln 2} \int_{e^y}^2 (x-1)\sqrt{1+e^2y}\, dx\, dy &= \int_0^{\ln 2} \sqrt{1+e^{2y}} \left(\int_{e^y}^2 (x-1)\, dx \right) dy \\
&= \int_0^{\ln 2} \sqrt{1+e^{2y}} \left[\frac{x^2}{2} - x \right]_{e^y}^2 dy \\
&= -\int_0^{\ln 2} \left(\frac{e^{2y}}{2} - e^y \right) \sqrt{1+e^{2y}}\, dy \\
&= -\frac{1}{2} \int_0^{\ln 2} e^{2y}\sqrt{1+e^{2y}}\, dy + \int_0^{\ln 2} e^y \sqrt{1+e^{2y}}\, dy\,. \qquad (3.5)
\end{aligned}$$

In the first integral in (3.5), we substitute

$$u = e^{2y}, \qquad du = 2e^{2y}\,dy,$$

and in the second integral we substitute

$$v = e^{y}, \qquad dv = e^{y}\,dy\,.$$

This gives

$$-\frac{1}{4}\int_1^4 \sqrt{1+u}\,du + \int_1^2 \sqrt{1+v^2}\,dv\,.$$

Both of these integrals can be computed using methods from single-variable calculus. (For instance, the antiderivatives are included in standard lists.) For the first integral, we have

$$\int_1^4 \sqrt{1+u}\,du = \left[\frac{2}{3}(1+u)^{3/2}\right]_1^4 = \frac{2}{3}\left(5^{3/2} - 2^{3/2}\right).$$

The second integral is

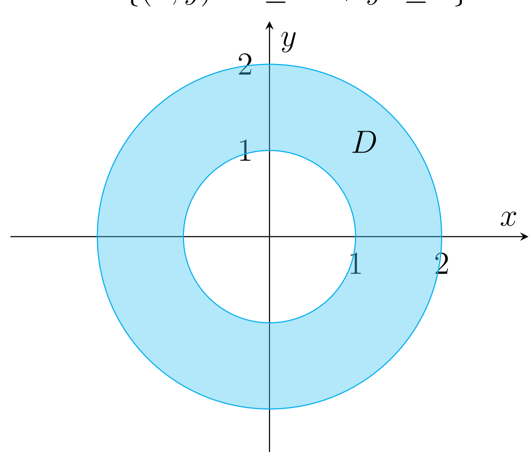
$$\begin{aligned}\int_1^2 \sqrt{1+v^2}\,dv &= \frac{1}{2}\left[v\sqrt{1+v^2} + \ln(\sqrt{1+v^2}+v)\right]_1^2 \\ &= \frac{1}{2}\left(2\sqrt{5} + \ln(\sqrt{5}+2)\right) - \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right).\end{aligned}$$

All together, the integral (3.4) is equal to

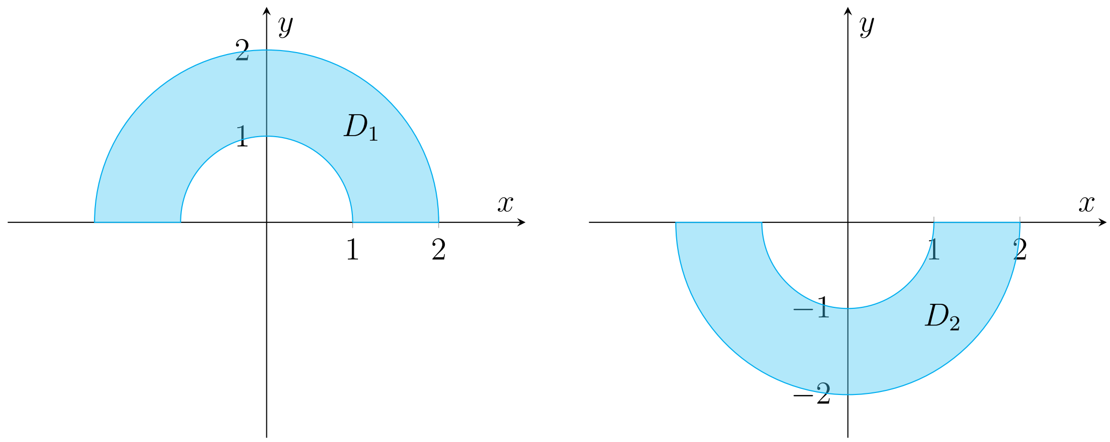
$$\begin{aligned}&-\frac{1}{4}\cdot\frac{2}{3}(5^{3/2} - 2^{3/2}) + \frac{1}{2}\left(2\sqrt{5} + \ln(\sqrt{5}+2)\right) - \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right). \\ &\qquad = \frac{1}{2}\left(2\sqrt{5} - \sqrt{2} + \ln\frac{\sqrt{5}+2}{\sqrt{2}+1}\right) - \frac{1}{6}\left(5^{3/2} - 2^{3/2}\right). \quad \triangle\end{aligned}$$

Example 3.6.4. How could we compute an integral over the region

$$D = \{(x,y) : 1 \le x^2 + y^2 \le 2\}?$$



The region D is neither x-simple nor y-simple. However, we can use additivity of the integral (Proposition 3.5.3) and split D into two y-simple regions:



Alternatively, we could split D into two x-simple regions. △

Exercises.

Exercises from [FRYb, §3.1]: Q4–Q29.

3.7 Triple integrals

Having discussed double integrals in some depth, we now turn our attention to higher dimensional integrals. The theory is analogous to the two-variable case and so we will be rather brief here. Although one can define multiple integrals in any finite dimension, we restrict our attention to triple integrals.

Our goal is to define the triple integral of a function over a box (more precisely, a rectangular parallelepiped)

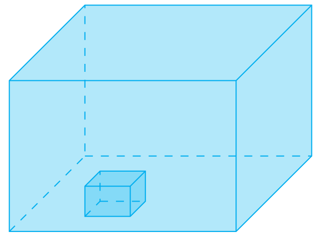
$$B = [a, b] \times [c, d] \times [p, q].$$

As for double integrals, we partition the three sides of the box into equal parts by subdividing the intervals

$$\begin{aligned} a = x_0 < x_1 < \cdots < x_n = b, \qquad & x_{i+1} - x_i = \Delta x := \frac{b-a}{n}, \\ c = y_0 < y_1 < \cdots < y_n = d, \qquad & y_{j+1} - y_j = \Delta y := \frac{d-c}{n}, \\ p = z_0 < z_1 < \cdots < z_n = q, \qquad & z_{k+1} - z_k = \Delta z := \frac{q-p}{n}. \end{aligned}$$

In this way we partition the box B into sub-boxes

$$B_{ijk} := [x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}], \qquad 0 \le i, j, k \le n-1.$$



Then we form the sum

$$S_n = \sum_{i,j,k=0}^{n-1} f(\mathbf{c}_{ijk})\Delta V,$$

where $\mathbf{c}_{ijk}$ is any point in B_{ijk} and

$$\Delta V = \Delta x \Delta y \Delta z$$

is the volume of each sub-box B_{ijk}.

Definition 3.7.1 (Triple integral)**.** Suppose that $B = [a, b] \times [c, d] \times [p, q] \subseteq \mathbb{R}^3$ and that f is a bounded function $f \colon B \to \mathbb{R}$. If $\lim_{n\to\infty} S_n = S$ exists and is independent of the choice of the points $\mathbf{c}_{ijk}$, then we say f is *integrable* and we call S the *triple integral* (or simple the *integral*) of f over B. We denote it by

$$\iiint\limits_B f\,dV, \qquad \iiint\limits_B f(x, y, z)\,dV, \qquad \text{or} \qquad \iiint\limits_B f(x, y, z)\,dx\,dy\,dz\,.$$

The triple integral satisfies the properties of linearity, homogeneity, monotonicity, and additivity. (See Propositions 3.3.4 and 3.5.3.) As for the double integral, we compute the triple integral in practice by reducing it to an iterated integral.

Theorem 3.7.2. *Suppose that $B = [a, b] \times [c, d] \times [p, q] \subseteq \mathbb{R}^3$ and $f \colon B \to \mathbb{R}$ is integrable. Then any iterated integral that exists is equal to the triple integral. For example,*

$$\begin{aligned} \iiint\limits_B f(x, y, z)\,dV &= \int_p^q \int_c^d \int_a^b f(x, y, z)\,dx\,dy\,dz \\ &= \int_p^q \int_a^b \int_c^d f(x, y, z)\,dy\,dx\,dz \\ &= \int_a^b \int_c^d \int_p^q f(x, y, z)\,dz\,dy\,dx\,. \end{aligned}$$

(There are six possible iterated integrals.)

Example 3.7.3. Let's compute

$$\iiint\limits_B (2x + y^2 - z)\,dV \quad \text{where} \quad B = [-1, 1] \times [0, 1] \times [0, 3].$$

Using Theorem 3.7.2, we have

$$\begin{aligned}
\iiint\limits_B (2x+y^2-z)\,dV &= \int_0^3 \int_0^1 \int_{-1}^1 (2x+y^2-z)\,dx\,dy\,dz \\
&= \int_0^3 \int_0^1 \left[x^2+(y^2-z)x\right]_{x=-1}^1 dy\,dz \\
&= 2\int_0^3 \int_0^1 (y^2-z)\,dy\,dz \\
&= 2\int_0^3 \left[\frac{y^3}{3}-zy\right]_{y=0}^1 dz \\
&= 2\int_0^3 \left(\frac{1}{3}-z\right) dz \\
&= 2\left[\frac{z}{3}-\frac{z^2}{2}\right]_{z=0}^3 = 2\left(1-\frac{9}{2}\right) = -7. \qquad \triangle
\end{aligned}$$

As for double integrals, we often want to integrate over more general regions W, not just boxes. As we did in Section 3.5, we do this by enclosing the region in a box and extending our function by zero outside of W.

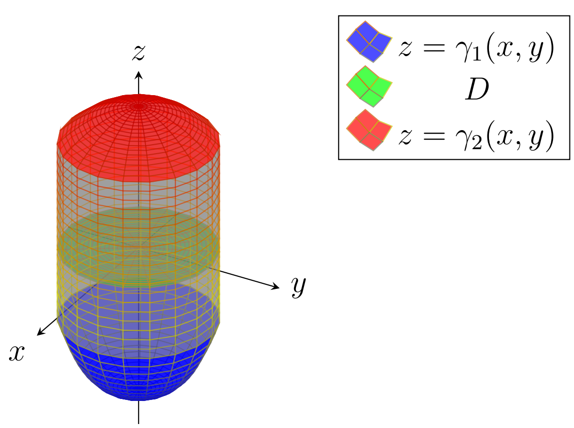
An *elementary region* in $\mathbb{R}^3$ is one defined by restricting one of the variables to lie between two functions of the remaining variables, and where the domain of these two functions is an elementary region in the plane. For example, if $D \subseteq \mathbb{R}^2$ is an elementary region and

$$\gamma_1, \gamma_2 \colon D \to \mathbb{R}, \qquad \gamma_1(x,y) \le \gamma_2(x,y) \text{ for all } (x,y) \in D,$$

then

$$W = \{(x,y,z) : (x,y) \in D,\ \gamma_1(x,y) \le z \le \gamma_2(x,y)\}$$

is an elementary region in $\mathbb{R}^3$.



Example 3.7.4. Let's describe the unit ball

$$W = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$$

as an elementary region.

First note that we can write

$$W = \left\{(x, y, z) : (x, y) \in D,\ -\sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2}\right\},$$

where

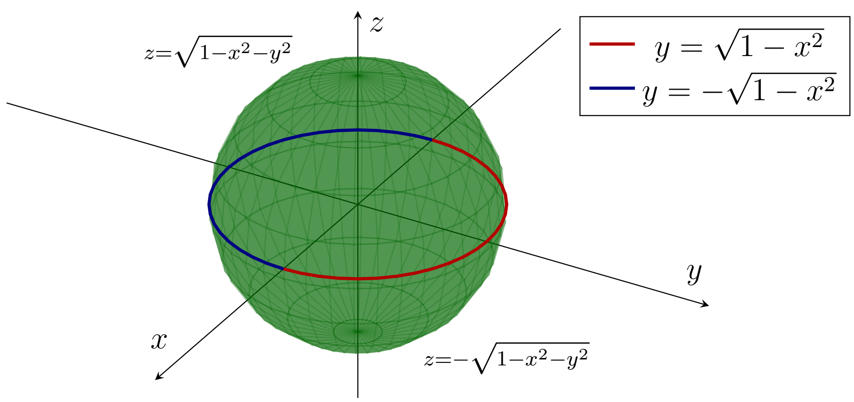
$$D = \{(x, y) : x^2 + y^2 \leq 1\}$$

is the unit disk. We can then describe D as a x-simple or as a y-simple region. As a y-simple region, it is

$$D = \{(x, y) : -1 \leq x \leq 1,\ -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}\}.$$

So we have described D by the inequalities

$$\begin{gathered} -1 \leq x \leq 1, \\ -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}, \\ -\sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2}. \end{gathered}$$



△

Theorem 3.7.5 (Triple iterated integrals)**.** *Suppose $D \subseteq \mathbb{R}^2$ is an elementary region,*

$$\gamma_1, \gamma_2 \colon D \to \mathbb{R}, \qquad \gamma_1(x, y) \leq \gamma_2(x, y) \text{ for all } (x, y) \in D,$$

and

$$W = \{(x, y, z) : (x, y) \in D,\ \gamma_1(x, y) \leq z \leq \gamma_2(x, y)\}.$$

Then

$$\iiint\limits_W f(x, y, z)\, dV = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

if

$$D = \{(x, y) : a \leq x \leq b,\ \phi_1(x) \leq y \leq \phi_2(x)\}$$

is a y-simple region, and

$$\iiint\limits_W f(x, y, z)\, dV = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)\, dz\, dx\, dy$$

if

$$D = \{(x, y) : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$$

is an x-simple region, provided all these integrals exist.

There are analogues of Theorem 3.7.5 for other orders of the variables.

Example 3.7.6. In the setting of Theorem 3.7.5, suppose $f(x, y, z) = 1$ for all $(x, y, z) \in W$. Then

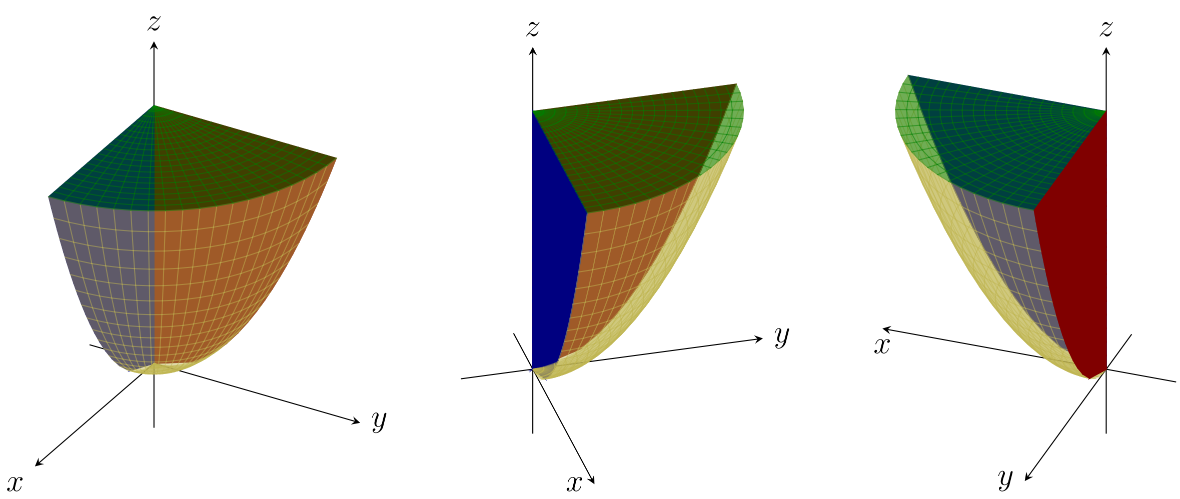
$$\iiint_W dV = \iint_D (\gamma_2(x, y) - \gamma_1(x, y))\, dA$$

is the volume of W. (Compare to (3.3).) △

Often the hardest part of computing a triple integral is finding the limits of integration.

Example 3.7.7 (cf. [MT11, Example 5.5.5]). Let W be the region bounded by the planes $x = 0$, $y = 0$, and $z = 2$, and the surface $z = x^2 + y^2$, and lying in the quadrant $x \ge 0$, $y \ge 0$. Let's compute the integral $\iiint_W y\, dx\, dy\, dz$.

We draw the region W from three different perspectives:



We can describe this region as

$$W = \{(x, y, z) : 0 \le x \le \sqrt{2},\ 0 \le y \le \sqrt{2 - x^2},\ x^2 + y^2 \le z \le 2\}.$$

Thus we have

$$\begin{aligned} \iiint_W y\, dx\, dy\, dz &= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} y\, dz\, dy\, dx \\ &= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} [yz]_{z=x^2+y^2}^{2}\, dy\, dx \\ &= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} y(2 - x^2 - y^2)\, dy\, dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\sqrt{2}} \left[y^2 - \frac{x^2 y^2}{2} - \frac{y^4}{4} \right]_{y=0}^{\sqrt{2-x^2}} dx \\
&= \int_0^{\sqrt{2}} \left(2 - x^2 - \frac{1}{2} x^2 (2 - x^2) - \frac{1}{4} (2 - x^2)^2 \right) dx \\
&= \int_0^{\sqrt{2}} \left(\frac{1}{4} x^4 - x^2 + 1 \right) dx \\
&= \left[\frac{x^5}{20} - \frac{x^3}{3} + x \right]_{x=0}^{\sqrt{2}} = \frac{\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} + \sqrt{2} = \frac{8\sqrt{2}}{15}.
\end{aligned}
\qquad \triangle
$$

Exercises.

Exercises from [FRYb, §3.5]: Q1–Q22.

Chapter 4

Change of variables

In this chapter we discuss changing variables in double and triple integrals. This is a very useful technique and one can sometimes greatly simplify an integral by changing to more suitable variables.

4.1 Two variables

Recall the change of variable formula, also called *substitution*, for single-variable integrals:

$$\int_a^b f(x)\,dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(u))\varphi'(u)\,du, \tag{4.1}$$

where $\varphi\colon [c,d] \to [a,b]$ is a C^1 one-to-one function. We often write

$$x = \varphi(u), \qquad dx = \varphi'(u)\,du,$$

to remember the extra factor of φ' that we need to include. We made these types of change of variables in Example 3.6.3. Sometime we use the change of variables formula in the other direction:

$$\int_a^b f(\varphi(x))\varphi'(x)\,dx = \int_{\varphi(a)}^{\varphi(b)} f(u)\,du\,.$$

For example, to compute $\int_0^{\sqrt{\pi/2}} x\cos(x^2)\,dx$, we can make the substitution

$$u = \varphi(x) = x^2, \qquad du = 2x\,dx,$$

so that

$$\int_0^{\sqrt{\pi/2}} x\cos(x^2)\,dx = \frac{1}{2}\int_0^{\pi/2} \cos u\,du = \frac{1}{2}\left[\sin u\right]_0^{\pi/2} = \frac{1}{2}.$$

In an iterated integral $\int_a^b \int_c^d f(x,y)\,dy\,dx$, we could make such a change of variables for x and y independently. However, we often want to change x and y *simultaneously*, in such a way that the new variables u and v are functions of *both* x and y:

$$(u,v) = T(x,y).$$

Since our definition of the double integral (Definition 3.3.1) involved dividing a region into rectangles, the key to developing the change of variables formula in two dimensions is seeing how the change of variables affects the area of a rectangle. In fact, it is enough see how it affects the area of the unit square.

Recall that any linear map $\mathbb{R}^2 \to \mathbb{R}^2$ corresponds to left multiplication by a 2×2 matrix. Consider the matrix

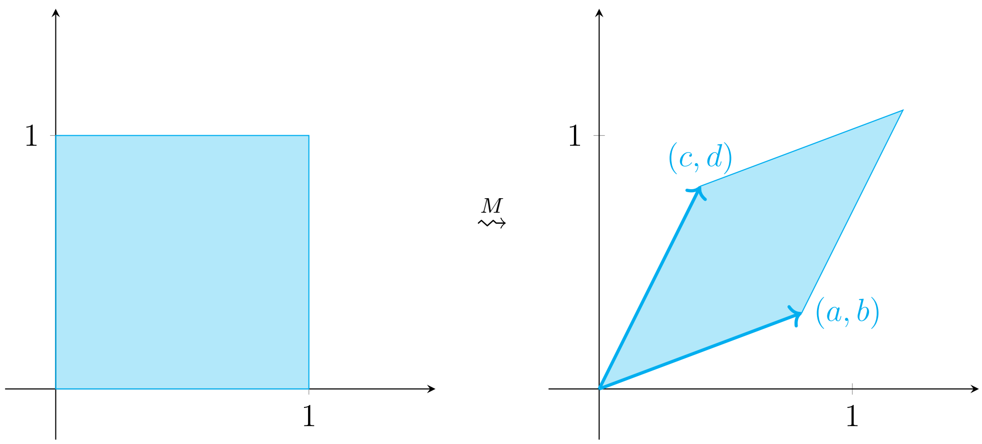
$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}).$$

This sends the unit square $[0,1] \times [0,1]$, which has vertices

$$(0,0), \quad (1,0), \quad (0,1), \quad (1,1),$$

to the parallelogram with vertices

$$(0,0), \quad (a,c), \quad (b,d), \quad (a+b, c+d).$$



The area of this parallelogram is the absolute value of

$$ad - bc = \det M,$$

the determinant of M.

In general, our change of variables may not be given by a linear map. However, we can approximate it by a linear map. Suppose our change of variables is given by

$$(x,y) = T(u,v), \quad \text{where } T\colon D^* \to \mathbb{R}^2 \text{ is } C^1.$$

We learned in Section 1.5 that the best linear approximation to the function T is given by its *Jacobian matrix*

$$DT = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}.$$

The determinant of this matrix is called the *Jacobian determinant*, or simply the *Jacobian*, of T. It is denoted

$$\frac{\partial(x,y)}{\partial(u,v)} := \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}.$$

In the limit defining the two-variable integral, this approximation becomes better and better. So when we change variables in two dimensions, we need to adjust by a factor equal to the absolute value of the Jacobian, to correct for the change in area induced by the change of variables. Recall that a function is *bijective* if it is injective (one-to-one) and surjective (onto).

Theorem 4.1.1 (Change of variables for double integrals)**.** *Suppose that D and D^* are elementary regions in the plane, and that*

$$T\colon D^* \to D, \quad (u,v) \mapsto (x(u,v), y(u,v)),$$

is bijective and of class C^1. Then, for any integrable function $f\colon D \to \mathbb{R}$, we have

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv,$$

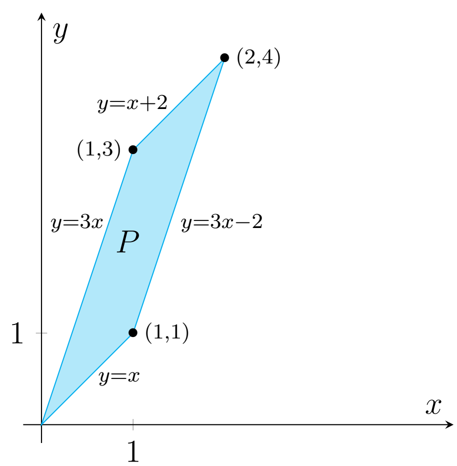
where $(x,y) = T(u,v)$.

We can remember the formula in Theorem 4.1.1 by

$$dA = dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv\,. \tag{4.2}$$

Example 4.1.2. Let P be the parallelogram bounded by

$$y = 3x, \quad y = x, \quad y = x+2, \quad y = 3x-2.$$



Suppose we want to compute the integral

$$\iint\limits_P xy\,dx\,dy\,.$$

Since P is an elementary region, we could compute this as an iterated integral. However, this would require breaking the region P into pieces. Alternatively, we can change variables in such a way that the region becomes much easier to work with.

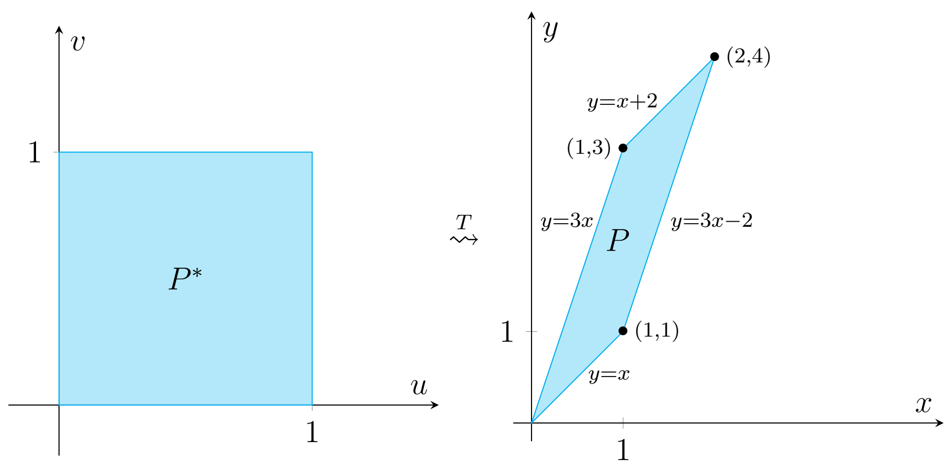
Let's make the change of variables

$$x = u + v, \quad y = 3u + v, \qquad \text{so that } T(u, v) = (u + v, 3u + v).$$

This T is the linear map given by the matrix

$$\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}.$$

Since the determinant of this matrix is -2, which is nonzero, it is one-to-one (as you learned in linear algebra). Thus we can use Theorem 4.1.1. We chose T so that it maps the unit square P^* to P:



Furthermore

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \left|\det \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}\right| = |-2| = 2.$$

Thus, we have

$$dA = dx\, dy = 2\, du\, dv\, .$$

Therefore, by the change of variables formula (Theorem 4.1.1), we have

$$\iint\limits_P xy\, dx\, dy = \iint\limits_{P^*} (u + v)(3u + v)2\, du\, dv = 2\int_0^1 \int_0^1 (3u^2 + 4uv + v^2)\, du\, dv$$

$$= 2\int_0^1 \left[u^3 + 2u^2 v + uv^2\right]_{u=0}^1 dv = 2\int_0^1 (1 + 2v + v^2)\, dv = 2\left[v + v^2 + \frac{v^3}{3}\right]_{v=0}^1 = \frac{14}{3}. \quad \triangle$$

Exercises.

4.1.1. The one-variable analogue of Theorem 4.1.1 recovers (4.1) in the following sense. Suppose $a < b$, $c < d$ and let

$$I = [a, b], \quad J = [c, d]$$

Furthermore, suppose that $\varphi\colon [c, d] \to [a, b]$ is a C^1 bijective function. Show that the formula

$$\int_a^b f(x)\, dx = \int_c^d f(\varphi(u))|\varphi'(u)|\, du$$

recovers the single-variable change of variable formula (4.1). The subtle point here is the presence of the absolute value above. *Hint*: The assumptions on φ imply that it is monotonic.

4.1.2 ([MT11, §6.2, Exercises 1, 2]). For each of the following integrals, suggest a change of variables that will simplify the integrand. Then find the corresponding Jacobian determinant.

(a) $\iint\limits_R (3x + 2y)\sin(x - y)\, dA$

(b) $\iint\limits_R e^{-4x+7y}\cos(7x - 2y)\, dA$

(c) $\iint\limits_R (5x + y)^3(x + 9y)^4\, dA$

(d) $\iint\limits_R \big(x\sin(6x + 7y) - 3y\sin(6x + 7y)\big)\, dA$

4.1.3 ([MT11, §6.2, Exercise 4]). Let D be the region $0 \le y \le x$ and $0 \le x \le 1$. Evaluate

$$\iint\limits_D (x + y)\, dx\, dy$$

by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly (i.e. without the change of variables) using an iterated integral.

4.1.4 ([MT11, §6.2, Exercise 8]). Define $T(u, v) = (u^2 - v^2, 2uv)$. Let

$$D^* = \{(u, v) : u^2 + v^2 \le 1,\ u \ge 0,\ v \ge 0\}.$$

Find $D = T(D^*)$ and evaluate $\iint\limits_D dx\, dy$.

4.1.5 ([MT11, §6.2, Exercise 10]). Calculate

$$\iint\limits_R \frac{1}{x + y}\, dy\, dx,$$

where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$, $x + y = 4$, by using the mapping $T(u, v) = (u - uv, uv)$.

4.1.6 ([MT11, §6.2, Exercise 14]). (a) Express $\int_0^1 \int_0^{x^2} xy\,dy\,dx$ as an integral over the triangle

$$D^* = \{(u, v) : 0 \le u \le 1,\ 0 \le v \le u\}.$$

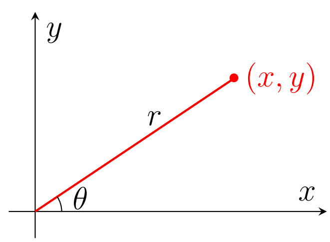
Hint: Find a one-to-one mapping T of D^* onto the given region of integration.

(b) Evaluate this integral directly and as an integral over D^*.

4.2 Polar coordinates

As a special case of the change of variables technique of Section 4.1, we now develop a method for computing integrals in polar coordinates. A good reference for the material in this section is [FRYc, §3.2].

Definition 4.2.1 (Polar coordinates). The *polar coordinates* of the point $(x, y) \in \mathbb{R}^2$ is the ordered pair (r, θ), where r is the distance from $(0, 0)$ to (x, y) and θ is the counter-clockwise angle between the x-axis and the line joining (x, y) to $(0, 0)$.



So we have

$$x = r\cos\theta, \qquad y = r\sin\theta,$$
$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\frac{y}{x}.$$

We can apply the general theory of Section 4.1. Taking

$$(x, y) = T(r, \theta) = (r\cos\theta, r\sin\theta), \tag{4.3}$$

we have

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det DT = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Note that T is not one-to-one, since it sends all $(0, \theta)$ to $(0, 0)$. However, the change of variables theorem is still valid. This is because the set of points where T is not one-to-one lies on the graph of a smooth curve (in particular, it is a one-dimensional region with zero area) and hence can be neglected for the purposes of integration.

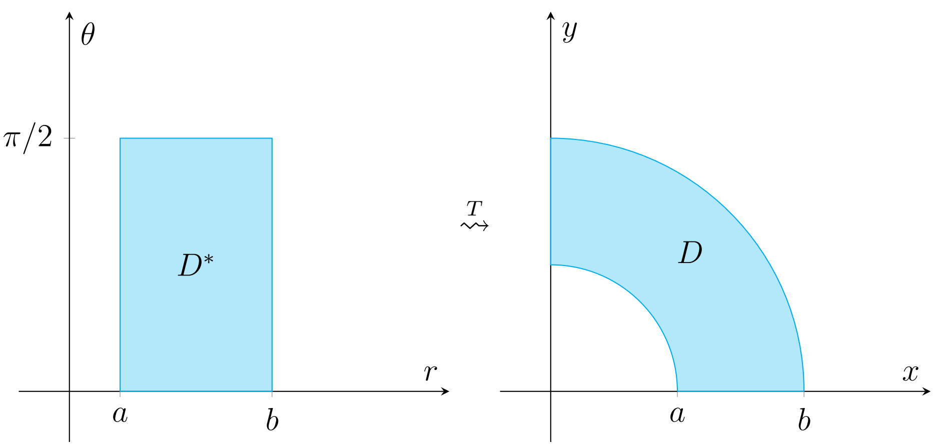
Theorem 4.2.2 (Double integral in polar coordinates). *Suppose that D is an elementary region and that the map T of (4.3) gives a bijection from D^* to D, except possibly for points on the boundary of D^*. Then*

$$\iint_D f(x, y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta\,.$$

Example 4.2.3. Let's evaluate

$$\iint\limits_D \ln(x^2+y^2)\,dx\,dy,$$

where D is the region in the first quadrant lying between the arcs of the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$, where $0<a<b$.



In polar coordinates, the region D corresponds to

$$D^* = \{(r,\theta) : a \le r \le b, \quad 0 \le \theta \le \tfrac{\pi}{2}\} = [a,b] \times \left[0, \tfrac{\pi}{2}\right].$$

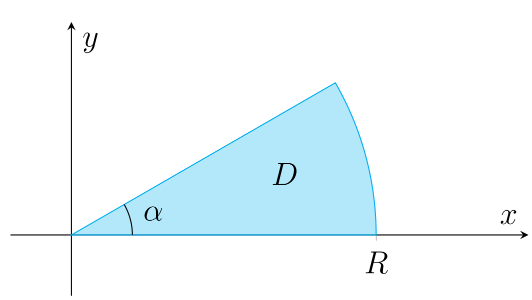
Therefore, by Theorem 4.2.2, we have

$$\iint\limits_D \ln(x^2+y^2)\,dx\,dy = \int_a^b \int_0^{\pi/2} r\ln(r^2)\,d\theta\,dr = \frac{\pi}{2}\int_a^b r\ln(r^2)\,dr = \pi\int_a^b r\ln r\,dr\,.$$

Using integration by parts, we obtain

$$\pi\left[\frac{r^2}{2}\ln r - \frac{r^2}{4}\right]_{r=a}^{b} = \frac{\pi}{4}\left(2b^2\ln b - 2a^2\ln a + a^2 - b^2\right). \qquad \triangle$$

Example 4.2.4. Let's compute the area of a sector of a circle of radius R and angle α.



In polar coordinates, the region is given by

$$0 \le r \le R, \qquad 0 \le \theta \le \alpha.$$

So we have

$$\text{Area}(D) = \iint_D dA = \int_0^R \int_0^\alpha r\, d\theta\, dr = \alpha \int_0^R r\, dr = \alpha \left[\frac{r^2}{2}\right]_{r=0}^R = \frac{1}{2}\alpha R^2.$$

In particular, the area of the circle of radius R (so $\alpha = 2\pi$) is πR^2. $\triangle$

Exercises.

Exercises from [FRYb, §3.2]: Q1–Q24.

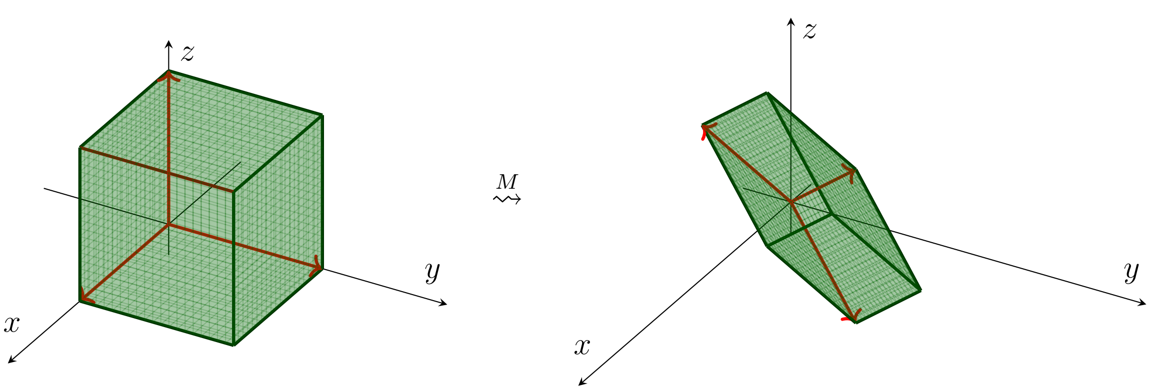
4.3 Three variables

We can generalize the theory of Section 4.1 to arbitrary dimensions. We covered two dimensions in Section 4.1 and one dimension in previous courses (see also Exercise 4.1.1). We now treat the case of three dimensions.

The situation for three dimensions is very similar to the two-dimensional case treated in Section 4.1. The key now is that the linear map $\mathbb{R}^3 \to \mathbb{R}^3$ given by left multiplication by the matrix $M \in \text{Mat}_{3\times 3}(\mathbb{R})$ sends the unit cube $[0,1]^3$ to the parallelepiped determined by the vectors $M\mathbf{e}_1, M\mathbf{e}_2, M\mathbf{e}_3$, where

$$\mathbf{e}_1 = (1,0,0), \quad \mathbf{e}_2 = (0,1,0), \quad \mathbf{e}_3 = (0,0,1)$$

are the standard basis vectors.



This parallelepiped has volume $|\det M|$.

Now suppose we introduce new variables u, v, w such that

$$(x,y,z) = T(u,v,w)$$

for some C^1 function

$$T \colon \mathbb{R}^3 \to \mathbb{R}^3.$$

As in Section 4.1, we use the fact that the best linear approximation to T is given by the Jacobian matrix DT. Therefore, the factor we need to add to our integral when changing coordinates is the absolute value of the Jacobian determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} := \det(DT) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}.$$

Theorem 4.3.1 (Change of variables for triple integrals)**.** *Suppose that W and W^* are elementary regions in $\mathbb{R}^3$, and that*

$$T \colon W^* \to W, \quad (u,v,w) \mapsto (x(u,v,w), y(u,v,w), z(u,v,w)),$$

is of class C^1 and is bijective, except possibly on a set that is a finite union of graphs of functions of two variables. Then, for any integrable function $f \colon W \to \mathbb{R}$, we have

$$\iiint\limits_W f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw\,.$$

Exercises.

4.3.1 ([MT11, §6.2, Exercise 20]). Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

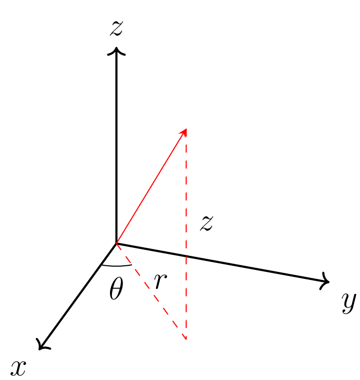
$$T(u,v,w) = (u\cos v \cos w, u \sin v \cos w, u \sin w).$$

(a) Show that T is onto the unit sphere; that is, every (x,y,z) with $x^2+y^2+z^2 = 1$ can be written as $(x,y,z) = T(u,v,w)$ for some (u,v,w).

(b) Show that T is not one-to-one.

4.4 Cylindrical coordinates

Let's apply the general change of variables formula for triple integrals (Theorem 4.3.1) to cylindrical coordinates. A good reference for the material in this section is [FRYc, §3.6].



Cylindrical coordinates, denoted (r,θ,z), of a point $(x,y,z) \in \mathbb{R}^3$, are defined by:

- r is the distance from $(x, y, 0)$ to $(0, 0, 0)$, which is the same as the distance from (x, y, z) to the z-axis;
- θ is the angle between the positive x-axis and the line joining $(x, y, 0)$ to $(0, 0, 0)$;
- z is the signed distance from (x, y, z) to the xy-plane (that is, it is the same z as in cartesian coordinates).

In other words, (r, θ) are polar coordinates in the xy-plane, and z is the usual z.

Cartesian and cylindrical coordinates are related by:

$$\begin{aligned} x &= r\cos\theta, & y &= r\sin\theta, & z &= z, \\ r &= \sqrt{x^2+y^2}, & \theta &= \arctan\tfrac{y}{x}, & z &= z. \end{aligned}$$

So we consider the change of variables map

$$(x, y, z) = T(r, \theta, z) = (r\cos\theta, r\sin\theta, z). \tag{4.4}$$

The Jacobian matrix of T is

$$DT = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Its determinant is

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r(\cos^2\theta + \sin^2\theta) = r.$$

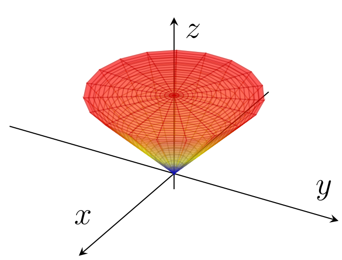
Applying Theorem 4.3.1 to this change of variables, we obtain the following result.

Theorem 4.4.1 (Triple integral in cylindrical coordinates)**.** *Suppose that W is an elementary region in $\mathbb{R}^3$ and that the map T of (4.4) gives a bijection from W^* to W, except possibly for points on the boundary of W^*. Then*

$$\iiint_W f(x, y, z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z) r\,dr\,d\theta\,dz\,.$$

Example 4.4.2. Consider the cone

$$W = \{(x, y, z) : z \geq 0,\ x^2 + y^2 \leq z^2 \leq 1\}.$$



Suppose the density of the cone at point (x, y, z) is equal to z. What is the mass of the cone?

In cylindrical coordinates, the cone is given by

$$W^* = \{(r,\theta,z) : 0 \le z \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le r \le z\}$$

Thus, we have

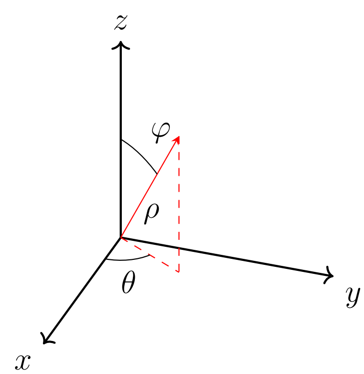
$$\begin{aligned}\text{Mass} &= \iiint_W z\,dV = \iiint_{W^*} rz\,dr\,d\theta\,dz = \int_0^1 \int_0^z \int_0^{2\pi} rz\,d\theta\,dr\,dz \\ &= 2\pi \int_0^1 \int_0^z rz\,dr\,dz = 2\pi \int_0^1 \left[\frac{r^2 z}{2}\right]_{r=0}^{z} dz = \pi \int_0^1 z^3\,dz = \pi \left[\frac{z^4}{4}\right]_{z=0}^{1} = \frac{\pi}{4}. \quad \triangle\end{aligned}$$

Exercises.

Exercises from [FRYb, §3.6]: Q1–Q17.

4.5 Spherical coordinates

Now let's apply the general change of variables formula for triple integrals (Theorem 4.3.1) to spherical coordinates. A good reference for the material in this section is [FRYc, §3.7].



Spherical coordinates, denoted (ρ,θ,φ), of a point $(x,y,z) \in \mathbb{R}^3$ are defined by:

- ρ is the distance from (x,y,z) to $(0,0,0)$;
- θ is the angle between the positive x-axis and the line joining $(x,y,0)$ to $(0,0,0)$;
- φ is the angle between the positive z-axis and the line joining (x,y,z) to $(0,0,0)$.

The spherical coordinate θ is the same as the cylindrical coordinate θ, while the spherical coordinate φ is new. We generally restrict these coordinates to the ranges

$$\rho \ge 0, \quad 0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \pi.$$

Cartesian and spherical coordinates are related by

$$x = \rho \sin\varphi \cos\theta, \qquad y = \rho \sin\varphi \sin\theta, \qquad z = \rho\cos\varphi,$$
$$\rho = \sqrt{x^2+y^2+z^2}, \qquad \theta = \arctan\frac{y}{x}, \qquad \varphi = \arctan\frac{\sqrt{x^2+y^2}}{z}.$$

So we consider the change of variables map

$$(x,y,z) = T(\rho,\theta,\varphi) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi). \tag{4.5}$$

As we noted in Example 1.12.2, we have

$$DT = \begin{bmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix}.$$

and

$$\det(DT) = -\rho^2\sin\varphi.$$

Applying Theorem 4.3.1 to this change of variables, we obtain the following result.

Theorem 4.5.1 (Triple integral in spherical coordinates)**.** *Suppose that W is an elementary region in $\mathbb{R}^3$ and that the map T of (4.5) gives a bijection from W^* to W, except possibly for points on the boundary of W^*. Then*

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi\,.$$

Example 4.5.2 (Volume of a ball)*.* Consider the ball of radius R, centred at the origin:

$$B = \{(x,y,z) : x^2+y^2+z^2 \le R^2\}.$$

In spherical coordinates the ball is given by

$$B^* = \{(\rho,\theta,\varphi) : 0 \le r \le R,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}.$$

Therefore

$$\begin{aligned} \operatorname{Vol} B &= \iiint_B dx\,dy\,dz = \iiint_{B^*} \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_0^\pi\int_0^{2\pi}\int_0^R \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi \\ &= \int_0^\pi\int_0^{2\pi}\left[\frac{\rho^3}{3}\right]_{\rho=0}^R \sin\varphi\,d\theta\,d\varphi = \frac{R^3}{3}\int_0^\pi\int_0^{2\pi}\sin\varphi\,d\theta\,d\varphi = \frac{2\pi R^3}{3}\int_0^\pi \sin\varphi\,d\varphi \\ &= \frac{2\pi R^3}{3}\left[-\cos\varphi\right]_{\varphi=0}^\pi = \frac{4\pi R^3}{3}. \qquad \triangle \end{aligned}$$

Exercises.

Exercises from [FRYb, §3.7]: Q1–Q31.

Chapter 5

Vector fields

Many of the remaining topics in the course will involve vector fields. In this chapter we define vector fields and discuss some important related concepts. A good reference for the material in this chapter is [FRYe, §2.1–§2.3].

5.1 Definitions and first examples

Definition 5.1.1 (Vector field)**.** (a) A *vector field in the plane* (or *in two dimensions*) is a function

$$\mathbf{v} \colon D \to \mathbb{R}^2, \qquad D \subseteq \mathbb{R}^2.$$

We denote its components by $\mathbf{v}(x, y) = (v_1(x, y), v_2(x, y))$.

(b) A *vector field in space* (or *in three dimensions*) is a function

$$\mathbf{v} \colon W \to \mathbb{R}^3, \qquad W \subseteq \mathbb{R}^3.$$

We denote its components by $\mathbf{v}(x, y, z) = (v_1(x, y, z), v_2(x, y, z), v_3(x, y, z))$.

Examples 5.1.2. (a) If a moving fluid occupies some region $W \subseteq \mathbb{R}^3$ and the velocity of the fluid is $\mathbf{v}(x, y, z)$ at position (x, y, z), then $\mathbf{v}$ is called a *velocity field*.

(b) If there is a force $\mathbf{F}(x, y, z)$ at position (x, y, z), then $\mathbf{F}$ is called a *force field*.

△

We will sometimes denote the unit vectors in the plane by

$$\mathbf{i} = (1, 0), \qquad \mathbf{j} = (0, 1),$$

and the unit vectors in space by

$$\mathbf{i} = (1, 0, 0), \qquad \mathbf{j} = (0, 1, 0), \qquad \mathbf{k} = (0, 0, 1).$$

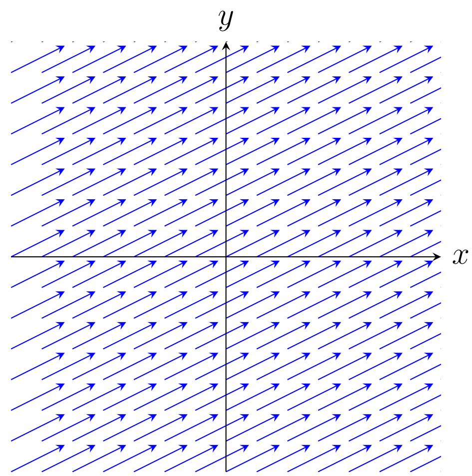
Note that we use the notation $\mathbf{i}$ to denote both $(1, 0)$ and $(1, 0, 0)$, and similarly for $\mathbf{j}$. The context should make clear which one we mean. Alternatively, if we view $\mathbb{R}^2$ as the subset $\{(x, y, 0) : x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$, then there is no ambiguity.

It is sometimes useful to sketch a vector field by drawing its values as arrows at a selection of points.

Example 5.1.3. The constant vector field

$$\mathbf{v}(x,y) = 2\mathbf{i} + \mathbf{j} = (2,1)$$

can be sketched as follows:

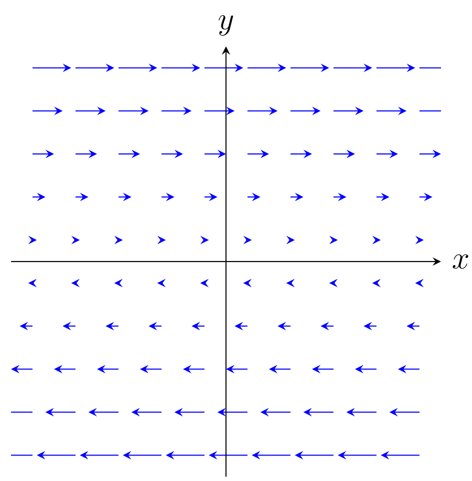


△

Example 5.1.4. The vector field

$$\mathbf{v}(x,y) = y\mathbf{i}$$

can be sketched as follows:

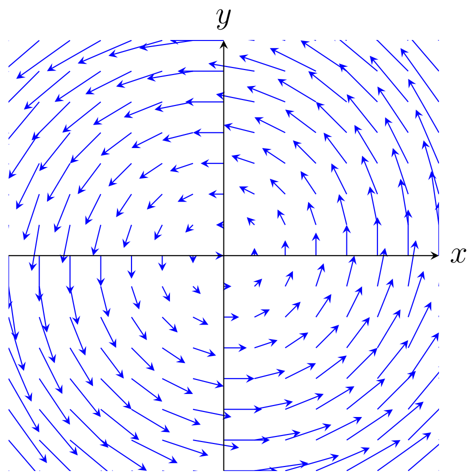


△

Example 5.1.5. The vector field

$$\mathbf{v}(x,y) = -y\mathbf{i} + x\mathbf{j} = (-y,x)$$

can be sketched as follows:



△

We will sometimes use the notation

$$\mathbf{r} = (x, y, z) \quad \text{and} \quad r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

to denote points in $\mathbb{R}^3$, and describe vector fields in terms of $\mathbf{r}$ and r.

Example 5.1.6 (Gravity). Suppose an object of mass M is placed at the origin $(0, 0, 0)$. Newton's law of gravitation states that the magnitude of the gravitational force experienced by a mass m at point $\mathbf{r} = (x, y, z)$ is given by $\frac{GMm}{r^2}$, where G is the gravitational constant and $r = \sqrt{x^2 + y^2 + z^2}$ is the distance to the origin (in other words, r is the distance between the two objects). This force points towards the origin. Thus, since $-\frac{\mathbf{r}}{r}$ is a unit vector pointing towards the origin, the gravitational force field for the object of mass m is given by

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^3}\mathbf{r}.$$

△

Exercises.

Exercises from [FRYd, §2.1]: Q1–Q15.

5.2 Conservative vector fields

As we will see in later chapters, some vector fields are considerably easier to work with than others. In this section, we examine one such type of vector field.

Recall, from (1.12), that the *gradient* of a differentiable function $f\colon A \to \mathbb{R}$, $A \subseteq \mathbb{R}^m$, is

$$\operatorname{grad} f = \nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}\right).$$

If $m = 2$, then we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j},$$

and, if $m = 3$, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

We sometimes remember this by writing

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial y}. \tag{5.1}$$

Definition 5.2.1 (Conservative vector field, potential, equipotential curve and surface)**.**

(a) A vector field $\mathbf{F}$ is said to be *conservative* if there exists a function φ such that

$$\mathbf{F} = \nabla\varphi.$$

In this case, φ is called a *potential* for $\mathbf{F}$. Note that if φ is a potential for $\mathbf{F}$ and $C \in \mathbb{R}$, then $\varphi + C$ is also a potential for $\mathbf{F}$.

(b) If $\mathbf{F} = \nabla\varphi$ is a conservative field in three dimensions with potential φ and $C \in \mathbb{R}$, then the set

$$\{(x, y, z) : \varphi(x, y, z) = C\}$$

is called an *equipotential surface*. Similarly, in two dimensions,

$$\{(x, y) : \varphi(x, y) = C\}$$

is called an *equipotential curve*.

Note that equipotential curves and surfaces are just level sets (see (1.15)). It follows from Proposition 1.8.6, that if $\mathbf{F} = \nabla\varphi$ is a conservative vector field, then the vector field is orthogonal to the equipotential surface or curve.

Warning 5.2.2. Physicists use a different convention for potentials. In physics, φ is called a potential for $\mathbf{F}$ when $\mathbf{F} = -\nabla\varphi$. Note the minus sign.

Example 5.2.3. Recall the gravitational force field of Example 5.1.6. This force is conservative, with potential

$$\varphi(\mathbf{r}) = \frac{GMm}{r}.$$

Indeed, we have

$$\begin{aligned}
\frac{\partial}{\partial x}\varphi(\mathbf{r}) &= \frac{\partial}{\partial x}\frac{GMm}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}\frac{GMm \cdot 2x}{(x^2+y^2+z^2)^{3/2}} = -\frac{GMm}{r^3}x, \\
\frac{\partial}{\partial y}\varphi(\mathbf{r}) &= \frac{\partial}{\partial y}\frac{GMm}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}\frac{GMm \cdot 2y}{(x^2+y^2+z^2)^{3/2}} = -\frac{GMm}{r^3}y, \\
\frac{\partial}{\partial z}\varphi(\mathbf{r}) &= \frac{\partial}{\partial z}\frac{GMm}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}\frac{GMm \cdot 2z}{(x^2+y^2+z^2)^{3/2}} = -\frac{GMm}{r^3}z.
\end{aligned}$$

Thus,

$$\nabla\varphi(\mathbf{r}) = -\frac{GMm}{r^3}(x, y, z) = -\frac{GMm}{r^3}\mathbf{r} = \mathbf{F}(\mathbf{r}). \qquad \triangle$$

Example 5.2.4. Consider the vector field

$$\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}.$$

Let's try to show this field is conservative and find a potential. A potential φ would have to satisfy

$$\frac{\partial \varphi}{\partial x}(x, y) = x \quad \text{and} \quad \frac{\partial \varphi}{\partial y}(x, y) = -y. \tag{5.2}$$

In order to satisfy the first equation in (5.2), we would need to have

$$\varphi(x, y) = \frac{x^2}{2} + \psi(y),$$

where $\psi(y)$ is a function of y only. For this to also satisfy the second equation in (5.2), we need

$$-y = \frac{\partial \varphi}{\partial y}(x, y) = \frac{\partial}{\partial y}\left(\frac{x^2}{2} + \psi(u)\right) = \psi'(y).$$

This holds if and only if

$$\psi(y) = -\frac{y^2}{2} + C, \qquad C \in \mathbb{R}.$$

Therefore,

$$\varphi(x, y) = \frac{x^2}{2} - \frac{y^2}{2} + C$$

is a potential for any $C \in \mathbb{R}$. △

We conclude this section with an example of a field that is *not* conservative.

Example 5.2.5. Consider the vector field

$$\mathbf{v}(x, y) = -y\mathbf{i} + x\mathbf{j}.$$

Let's prove by contradiction that $\mathbf{v}$ is not conservative. Thus, we begin by assuming that $\mathbf{v}$ *is* conservative (and then try to arrive at a contradiction). So $\mathbf{v} = \nabla\varphi$ for some differentiable function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$. Thus we have

$$\frac{\partial \varphi}{\partial x}(x, y) = -y \quad \text{and} \quad \frac{\partial \varphi}{\partial y}(x, y) = x.$$

Proceeding as in Example 5.2.4, we have

$$\varphi(x, y) = -xy + \psi(y)$$

for some function $\psi\colon \mathbb{R} \to \mathbb{R}$. Then we have

$$x = \frac{\partial \varphi}{\partial y}(x, y) = \frac{\partial}{\partial y}\left(-xy + \psi(y)\right) = -x + \psi'(y) \iff \psi'(y) = 2x.$$

But this is a contradiction, since $\psi(y)$ cannot depend on x. Hence, $\mathbf{v}$ is not conservative. △

Exercises.

Exercises are deferred until Section 5.3.

5.3 Curl

We now introduce the important notion of the curl of a vector field. In addition to being important in later chapters, we will see that curl gives an important test for conservative vector fields.

Definition 5.3.1 (Curl). The *curl* of a vector field $\mathbf{F}(x, y, z)$, is defined by

$$\begin{aligned}\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \\ &= \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3\end{bmatrix}.\end{aligned}$$

(The determinant expression is just a mnemonic device.) If $\nabla \times \mathbf{F} = \mathbf{0}$, we say that $\mathbf{F}$ is *curl free*.

Note that curl only applies to vector fields in three dimensions. However, any vector field $\mathbf{v}(x, y)$ in two dimensions can be extended to a vector field in three dimensions by setting

$$\tilde{\mathbf{v}}(x, y, z) = \mathbf{v}(x, y).$$

Then its curl is given by

$$\nabla \times \tilde{\mathbf{v}} = \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}. \tag{5.3}$$

We will sometimes simply reuse the notation $\mathbf{v}$ for the three-dimensional extension, writing $\nabla \times \mathbf{v}$ for its curl. The quantity

$$\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$$

is sometimes called the *scalar curl* of the two-dimensional vector field $\mathbf{v}$.

Examples 5.3.2. (a) Consider the gravitational force field

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{(x^2 + y^2 + z^2)^{3/2}}(x, y, z).$$

We have

$$\frac{\partial F_1}{\partial y} = \frac{3GMmxy}{(x^2 + y^2 + z^2)^{5/2}}, \qquad \frac{\partial F_1}{\partial z} = \frac{3GMmxz}{(x^2 + y^2 + z^2)^{5/2}},$$

$$\frac{\partial F_2}{\partial x} = \frac{3GMmxy}{(x^2+y^2+z^2)^{5/2}}, \qquad \frac{\partial F_2}{\partial z} = \frac{3GMmyz}{(x^2+y^2+z^2)^{5/2}},$$
$$\frac{\partial F_3}{\partial x} = \frac{3GMmxz}{(x^2+y^2+z^2)^{5/2}}, \qquad \frac{\partial F_3}{\partial y} = \frac{3GMmyz}{(x^2+y^2+z^2)^{5/2}}.$$

Thus $\nabla \times \mathbf{F} = \mathbf{0}$.

(b) Consider the vector field

$$\mathbf{v}(x, y, z) = -y\mathbf{i} + x\mathbf{j} = (-y, x, 0)$$

of Example 5.1.5 (extended to three dimensions). We have

$$\frac{\partial v_1}{\partial y} = -1 \quad \text{and} \quad \frac{\partial v_2}{\partial x} = 1.$$

Therefore

$$\nabla \times \mathbf{v} = 2\mathbf{k}.$$

△

Theorem 5.3.3 (Test for conservative vector fields)**.** (a) *Suppose that* $\mathbf{F}(x, y)$ *is a* C^2 *vector field in the plane. If* $\mathbf{F}$ *is conservative, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}.$$

(b) *Suppose that* $\mathbf{F}(x, y, z)$ *is a* C^2 *vector field in space. If* $\mathbf{F}$ *is conservative, then*

$$\nabla \times \mathbf{F} = \mathbf{0}.$$

Note that, using (5.3), both cases of Theorem 5.3.3 involve the condition $\nabla \times \mathbf{F} = \mathbf{0}$.

Proof. We give the proof of (b), since the proof of (a) is similar and easier (and can be found in [FRYe, Th. 2.3.9].) Suppose $\mathbf{F}$ is conservative. Then there exists a potential φ with $\mathbf{F} = \nabla\varphi$. In other words, we have

$$\frac{\partial \varphi}{\partial x} = F_1, \quad \frac{\partial \varphi}{\partial y} = F_2, \quad \frac{\partial \varphi}{\partial z} = F_3.$$

Using Clairaut's theorem (Theorem 1.9.6), we have

$$\frac{\partial F_3}{\partial y} = \frac{\partial^2 \varphi}{\partial y\,\partial z} = \frac{\partial^2 \varphi}{\partial z\,\partial y} = \frac{\partial F_2}{\partial z},$$
$$\frac{\partial F_1}{\partial z} = \frac{\partial^2 \varphi}{\partial z\,\partial x} = \frac{\partial^2 \varphi}{\partial x\,\partial z} = \frac{\partial F_3}{\partial x},$$
$$\frac{\partial F_2}{\partial x} = \frac{\partial^2 \varphi}{\partial x\,\partial y} = \frac{\partial^2 \varphi}{\partial y\,\partial x} = \frac{\partial F_1}{\partial y}.$$

Hence $\nabla \times \mathbf{F} = \mathbf{0}$. □

Example 5.3.4. In Example 5.3.2(b), we saw that the vector field $\mathbf{v}(x,y,z) = -y\mathbf{i} + x\mathbf{j}$ has

$$\frac{\partial v_1}{\partial y} = -1 \neq 1 = \frac{\partial v_2}{\partial x}.$$

Thus, by Theorem 5.3.3(a), we can conclude that $\mathbf{v}$ is *not* conservative. We also proved this directly in Example 5.2.5. △

Warning 5.3.5. The converse of Theorem 5.3.3 is false! If $\nabla \times \mathbf{F} \neq \mathbf{0}$, then you can use Theorem 5.3.3 to conclude that $\mathbf{F}$ is *not* conservative. However, if $\nabla \times \mathbf{F} = \mathbf{0}$, you *cannot* conclude that $\mathbf{F}$ is conservative without doing more work.

Example 5.3.6. Consider the vector field

$$\mathbf{v}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

Then

$$\frac{\partial v_1}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial v_2}{\partial x}.$$

However, $\mathbf{v}$ is *not* conservative. We will prove this in Example 6.4.4. △

We conclude this section with one final example.

Example 5.3.7 ([FRYe, Example 2.3.13]). Consider the vector field

$$\mathbf{F}(x,y,z) = (y^2 + 2xz^2 - 1)\mathbf{i} + (2x+1)y\mathbf{j} + (2x^2z + z^3)\mathbf{k}.$$

We'd like to determine whether or not this vector field is conservative and, if it is, find a potential.

We first apply the test for conservative vector fields (Theorem 5.3.3). We compute

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + 2xz^2 - 1 & (2x+1)y & 2x^2z + z^3 \end{bmatrix} = 0\mathbf{i} - (4xz - 4xz)\mathbf{j} + (2y - 2y)\mathbf{k} = \mathbf{0}.$$

So $\mathbf{F}$ passes the test. However, we must remember (Warning 5.3.5) that this, by itself, does not imply that $\mathbf{F}$ is conservative.

Let's try to find a potential φ. We need

$$\frac{\partial \varphi}{\partial x}(x,y,z) = y^2 + 2xz^2 - 1, \tag{5.4}$$

$$\frac{\partial \varphi}{\partial y}(x,y,z) = (2x+1)y, \tag{5.5}$$

$$\frac{\partial \varphi}{\partial z}(x,y,z) = 2x^2z + z^3. \tag{5.6}$$

From (5.4), we have

$$\varphi(x,y,z) = xy^2 + x^2z^2 - x + \psi(y,z).$$

This satisfies (5.5) if and only if

$$\begin{aligned}
&\frac{\partial}{\partial y}(xy^2 + x^2z^2 - x + \psi(y,z)) = (2x+1)y \\
&\iff 2xy + \frac{\partial \psi}{\partial y}(y,z) = (2x+1)y \\
&\iff \frac{\partial \psi}{\partial y}(y,z) = y \\
&\iff \psi(y,z) = \frac{y^2}{2} + \zeta(z).
\end{aligned}$$

Finally, this satisfies (5.6) if and only if

$$\begin{aligned}
&\frac{\partial}{\partial z}\left(xy^2 + x^2z^2 - x + \frac{y^2}{2} + \zeta(z)\right) = 2x^2z + z^3 \\
&\iff 2x^2z + \zeta'(z) = 2x^2z + z^3 \\
&\iff \zeta(z) = \frac{z^4}{4} + C,
\end{aligned}$$

for some constant $C \in \mathbb{R}$. Thus, one possible potential (the one with $C = 0$) is

$$\varphi(x,y,z) = xy^2 + x^2z^2 - x + \frac{y^2}{2} + \frac{z^4}{4}.$$

We can double-check our computations by computing the gradient:

$$\nabla\varphi(x,y,z) = (y^2 + 2xz^2 - 1)\mathbf{i} + (2x+1)y\mathbf{j} + (2x^2z + z^3)\mathbf{k} = \mathbf{F}(x,y,z). \qquad \triangle$$

Exercises.

Exercises from [FRYd, §2.3]: Q2–Q11.

Chapter 6

Path and line integrals

In this chapter we will see how to integrate along curves. There are two types of such integrals. We can integrate a *scalar field* or a *vector field*. The first gives what is called a *path integral*, while the second is called a *line integral*. A good reference for the material in this chapter is [FRYe, Ch. 1 and §2.4].

6.1 Path integrals

In this section we discuss the integration of a scalar field (just another name for real-valued function) along a curve.

Recall from Definition 1.7.1 that a *path* is a map

$$\mathbf{r}\colon [a,b] \to \mathbb{R}^n,$$

where $a, b \in \mathbb{R}$, $a \leq b$. (In Section 1.7, we used the notation $c(t)$. We will now start to use boldface notation, such as $\mathbf{r}(t)$, since we want to emphasize the vector nature of these paths.) Throughout this chapter, we will assume that paths are continuous. A *curve* is the image of a path:

$$C = \{\mathbf{r}(t) : a \leq t \leq b\}.$$

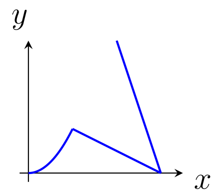
If $\mathbf{r}(t)$ is injective (one-to-one), we say that $\mathbf{r}(t)$ is a *parameterization* of the curve C. We also allow the possibility that $\mathbf{r}(a) = \mathbf{r}(b)$, in which case we say that C is a *closed curve*.

Definition 6.1.1 (Piecewise C^k)**.** We say a function $f\colon [a,b] \to \mathbb{R}^n$ is *piecewise* C^k if there exist real numbers $a = t_0 < t_1 < \dots < t_j = b$ such that the restrictions $f|_{[t_{i-1},t_i]}$ are C^k for $i = 1, \dots, j$.

Example 6.1.2. The function

$$f\colon [0,3] \to \mathbb{R}^2, \quad f(t) = \begin{cases} (t, t^2) & 0 \leq t \leq 1, \\ (2t-1, 2-t) & 1 \leq t \leq 2, \\ (5-t, 3t-6) & 2 \leq t \leq 3. \end{cases}$$

is piecewise C^1.



In fact, it is piecewise C^k for all k (hence piecewise C^∞). △

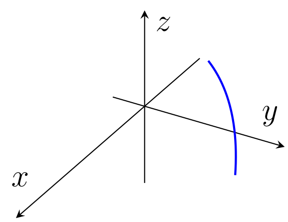
Example 6.1.3. A curve can have many different parameterizations. For instance,

$$\mathbf{r}\colon [0,1] \to \mathbb{R}^3, \quad \mathbf{r}(t) = (t^2, 1+t, 1-t)$$

and

$$\mathbf{s}\colon \left[0, \tfrac{1}{2}\right] \to \mathbb{R}^3, \quad \mathbf{s}(t) = ((1-2t)^2, 2-2t, 2t)$$

both parameterize the same curve.



△

Suppose that we have a curve C that is parameterized as $\mathbf{r}(t)$, $a \le t \le b$. Suppose furthermore that this curve is actually a wire whose density at the point $\mathbf{r}$ is $\rho(\mathbf{r})$. Let's try to compute the mass of this wire.

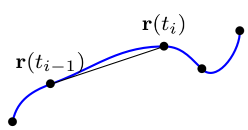
We divide the interval $[a,b]$ into n equal subintervals, each of length

$$\Delta t = \frac{b-a}{n}.$$

Let

$$t_i = a + i\Delta t$$

denote the endpoint of the i-th interval. We approximate the length of the part of the curve between $\mathbf{r}(t_{i-1})$ and $\mathbf{r}(t_i)$ by $\|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\|$ and the mass of this part of the curve by $\rho(\mathbf{r}(t_i))\|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\|$.



This gives us an approximate mass for the wire of

$$\sum_{i=1}^{n} \rho(\mathbf{r}(t_i))\|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\| = \sum_{i=1}^{n} \rho(\mathbf{r}(t_i)) \left\| \frac{\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})}{t_i - t_{i-1}} \right\| \Delta t.$$

We then take the limit as $n \to \infty$. If $\mathbf{r}(t)$ is piecewise C^1 and $\rho(\mathbf{r})$ is continuous, we get

$$\text{Mass of } C = \int_a^b \rho(\mathbf{r}(t)) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) \right\| dt\,.$$

Definition 6.1.4 (Path integral, integral along a curve)**.** Suppose C is a curve parameterized by

$$\mathbf{r}\colon [a,b] \to \mathbb{R}^n, \quad a \leq t \leq b,$$

and $f\colon A \to \mathbb{R}$ is a function, with $C \subseteq A \subseteq \mathbb{R}^n$. Then we define

$$\int_C f\, ds := \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\|\, dt\,. \tag{6.1}$$

In this notation, C specifies the curve, and ds denotes arc length. The integral (6.1) is called a *path integral*.

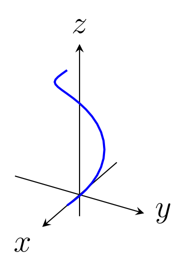
If $\mathbf{r}$ is piecewise C^1 and f is piecewise continuous, then the integral (6.1) always exists. Note that the notation $\int_C f\, ds$ only involves the curve C, and not the parameterizing path $\mathbf{r}$. We will justify this in Section 6.3.

Definition 6.1.5 (Length of a curve)**.** Suppose C is a curve parameterized by a piecewise C^1 path $\mathbf{r}\colon [a,b] \to \mathbb{R}^n$, $a \leq t \leq b$. Then the *length* of C is

$$\operatorname{Length}(C) := \int_C ds = \int_a^b \|\mathbf{r}'(t)\|\, dt\,.$$

Example 6.1.6. Fix $a, b \in \mathbb{R}$. Suppose we have a helical wire parameterized by

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = (a \cos t, a \sin t, bt), \quad 0 \leq t \leq 2\pi.$$



Furthermore, suppose the density at the point (x, y, z) of the wire is equal to z. Let's compute the length and the mass of the wire.

We have

$$\mathbf{r}'(t) = (-a \sin t, a \cos t, b),$$

and so

$$\|\mathbf{r}'(t)\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}.$$

Thus, the length of the wire is

$$\int_C ds = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{a^2+b^2}\, dt = (b-a)\sqrt{a^2+b^2}.$$

The mass of the wire is

$$\int_C bt \|\mathbf{r}'(t)\|\, dt = \int_a^b bt\sqrt{a^2+b^2}\, dt = \left[b\sqrt{a^2+b^2}\frac{t^2}{2} \right]_{t=a}^{b} = \tfrac{1}{2} b(b^2-a^2)\sqrt{a^2+b^2}. \qquad \triangle$$

Definition 6.1.7 (Average value along a curve)**.** The *average value* of a function f on a curve C is

$$\frac{\int_C f\,ds}{\text{Length}(C)}.$$

Example 6.1.8. Consider

$$\mathbf{r}(t) = (\sqrt{3}t^2, t^3 - t), \quad t \in [-1, 1].$$

Let's find the average x-value on the corresponding curve C. We have

$$\begin{aligned} \mathbf{r}'(t) &= (2\sqrt{3}t, 3t^2 - 1), \\ \|\mathbf{r}'(t)\|^2 &= (2\sqrt{3}t)^2 + (3t^2 - 1)^2 \\ &= 12t^2 + 9t^4 - 6t^2 + 1 \\ &= 9t^4 + 6t^2 + 1 \\ &= (3t^2 + 1)^2 \\ \|\mathbf{r}'(t)\| &= 3t^2 + 1. \end{aligned}$$

Therefore,

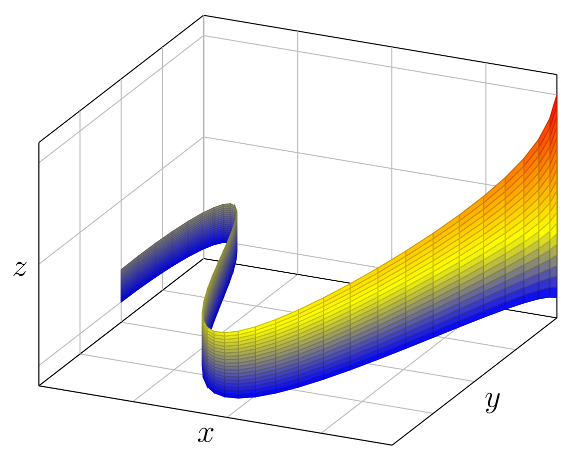
$$\text{Length}(C) = \int_C ds = \int_{-1}^1 (3t^2 + 1)\,dt = \left[t^3 + t\right]_{t=-1}^1 = 4$$

and

$$\begin{aligned} \int_C x\,ds = \int_{-1}^1 \sqrt{3}t^2(3t^2 + 1)\,dt = \sqrt{3}\int_{-1}^1 (3t^4 + t^2)\,dt = \sqrt{3}\left[\frac{3t^5}{5} + \frac{t^3}{3}\right]_{t=-1}^1 \\ = \sqrt{3}\left(\tfrac{6}{5} + \tfrac{2}{3}\right) = \frac{28\sqrt{3}}{15}. \end{aligned}$$

So the average x-value is $\frac{7\sqrt{3}}{15}$. △

We conclude this section with another intuitive interpretation of path integrals. Suppose $\mathbf{r}\colon [a, b] \to \mathbb{R}^2$ is a path with image C and $f\colon C \to \mathbb{R}$ is a function taking nonnegative values. So $f(\mathbf{r}(t)) \geq 0$ for all $t \in [a, b]$. Then the path integral $\int_C f\,ds$ is the surface area of a fence along the curve C whose height at point $\mathbf{r}(t)$ is given by $f(\mathbf{r}(t))$.



Exercises.

Exercises from [FRYd, §1.6]: Q1–Q11.

6.2 Line integrals

In Section 6.1 we saw how to integrate a *scalar* function along a curve. In this section, we will see a second type of integral along curves. This time we will integrate a *vector*-valued function along the curve.

If $\mathbf{r}\colon [a,b] \to \mathbb{R}^n$ is a parameterization of a curve C, we say that $\mathbf{F}$ is a *vector field on the path* $\mathbf{r}$ (or *on the curve* C) if $\mathbf{F}\colon A \to \mathbb{R}^n$ is a vector field with $C \subseteq A \subseteq \mathbb{R}^n$. In other words, the domain of $\mathbf{F}$ contains the curve C.

Definition 6.2.1 (Line integral)**.** Suppose $\mathbf{r}\colon [a,b] \to \mathbb{R}^n$ is a piecewise C^1 path, and $\mathbf{F}$ is a piecewise continuous vector field on $\mathbf{r}$. Then we define the *line integral*

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt\,.$$

If $\mathbf{r}(a) = \mathbf{r}(b)$, then the notation $\oint_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s}$ is also used. If $n = 3$ and $\mathbf{F} = (F_1, F_2, F_3)$, we also write

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{r}} (F_1\, dx + F_2\, dy + F_3\, dz).$$

If $n = 2$ and $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2)$, we sometimes write

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{r}} (F_1\, dx + F_2\, dy).$$

The notation $\mathbf{F} \cdot d\mathbf{s}$ in Definition 6.2.1 is meant to remind you of the dot product. If

$$d\mathbf{s} = (dx, dy, dy),$$

is an infinitesimal displacement, then $\mathbf{F} \cdot d\mathbf{s} = F_1\, dx + F_2\, dy + F_3\, dz$.

Example 6.2.2. Suppose

$$\mathbf{r}(t) = (t, t^2), \quad t \in [0,1] \quad \text{and} \quad \mathbf{F}(x,y) = (-y, x).$$

Let's compute $\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s}$.

We have

$$\mathbf{r}'(t) = (1, 2t)$$
$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(t, t^2) = (-t^2, t).$$

Therefore,

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (-t^2, t) \cdot (1, 2t) = -t^2 + 2t^2 = t^2.$$

Thus,

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 t^2\, dt = \left[\frac{t^3}{3}\right]_{t=0}^1 = \frac{1}{3}. \qquad \triangle$$

Definition 6.2.3 (Work)**.** If $\mathbf{F}(t)$ denotes a force moving a particle whose position is given by $\mathbf{r}(t)$, with t indicating time, then $\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s}$ is the *work* done by the force.

Example 6.2.4. Recall from Example 5.1.6 that the gravitational force field for an object of mass m in the presence of a mass M at the origin is given by

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^3}\mathbf{r},$$

where $\mathbf{r} = (x, y, z)$ is the position of the object of mass m. Suppose this object moves along a path

$$\mathbf{r}\colon [a, b] \to \mathbb{R}^3.$$

What work is done by gravity? By Definition 6.2.3,

$$\text{Work} = \int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = -\int_a^b \frac{GMm}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t) \cdot \mathbf{r}'(t)\, dt\,.$$

Let's make the substitution

$$u = \|\mathbf{r}(t)\|^2 = x(t)^2 + y(t)^2 + z(t)^2,$$

so that

$$du = 2x(t)x'(t)\, dt + 2y(t)y'(t)\, dt + 2z(t)z'(t)\, dt = 2\mathbf{r}(t) \cdot \mathbf{r}'(t)\, dt\,.$$

Thus

$$\text{Work} = -\int_{\|\mathbf{r}(a)\|^2}^{\|\mathbf{r}(b)\|^2} \frac{GMm}{2u^{3/2}}\, du = \left[\frac{GMm}{\sqrt{u}}\right]_{u=\|\mathbf{r}(a)\|^2}^{\|\mathbf{r}(b)\|^2} = \frac{GMm}{\|\mathbf{r}(b)\|} - \frac{GMm}{\|\mathbf{r}(a)\|} = \frac{GMm}{r_{\text{end}}} - \frac{GMm}{r_{\text{start}}},$$

where r_{start} is the distance from the origin at the start of the path and r_{end} is the distance from the origin at the end of the path. Note that the total work done only depends on the endpoints of the path, and not on the route taken in between or the time taken to travel between the points. We will explain this phenomenon in Example 6.4.2. △

Exercises.

Exercises are deferred until Section 6.5.

6.3 Reparameterization

Our definitions of path integrals (Definition 6.1.4) and line integrals (Definition 6.2.1) both involve a parameterization $\mathbf{r}(t)$ of a curve C. It is natural to ask how these integrals depend on the parameterization. We now examine this question.

Suppose

$$\mathbf{r}\colon [a, b] \to \mathbb{R}^n \quad \text{and} \quad \tilde{\mathbf{r}}\colon [c, d] \to \mathbb{R}^n$$

are two parameterizations of the same curve. In other words, they have the same image C. Since parameterizations are invertible, we have the composite function

$$g := \mathbf{r}^{-1} \circ \tilde{\mathbf{r}} \colon [c, d] \to [a, b]. \tag{6.2}$$

Thus

$$\tilde{\mathbf{r}}(u) = \mathbf{r}(g(u)), \quad u \in [c, d].$$

We call $\tilde{\mathbf{r}}$ a *reparameterization* of $\mathbf{r}$.

Since we assume that $\mathbf{r}$ and $\tilde{\mathbf{r}}$ are continuous, the map g is a continuous bijection. Therefore it is either increasing or decreasing. If g is increasing, we say that $\mathbf{r}$ and $\tilde{\mathbf{r}}$ have the *same orientation*, and that the reparameterization is *orientation-preserving*. If g is decreasing, we say that $\mathbf{r}$ and $\tilde{\mathbf{r}}$ have *opposite orientation*, and that the reparameterization is *orientation-reversing*.

Theorem 6.3.1 (Reparameterization for path integrals)**.** *Under the above hypotheses,*

$$\int_c^d f(\tilde{\mathbf{r}}(u)) \|\tilde{\mathbf{r}}'(u)\| \, du = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt \,.$$

Proof. This follows from the change of variables formula. Setting $t = g(u)$, we have

$$dt = g'(u) \, du, \qquad \tilde{\mathbf{r}}'(u) = \frac{\mathrm{d}}{\mathrm{d}u} \mathbf{r}(g(u)) = \mathbf{r}'(g(u)) g'(u).$$

If g is increasing, we have $g'(u) \ge 0$ for $u \in [a, b]$, $g(c) = a$, and $g(d) = b$. Thus,

$$\|\tilde{\mathbf{r}}'(u)\| = \|\mathbf{r}'(g(u)) g'(u)\| = \|\mathbf{r}'(g(u))\| g'(u).$$

Therefore,

$$\int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt = \int_c^d f(\mathbf{r}(g(u))) \|\mathbf{r}'(g(u))\| g'(u) \, du = \int_c^d f(\tilde{\mathbf{r}}(u)) \|\tilde{\mathbf{r}}'(u)\| \, du \,.$$

We leave it as an exercise (Exercise 6.3.1) to handle the case where g is decreasing. $\square$

Theorem 6.3.1 shows that, while we need to parameterize a curve in order to integrate along it, the integral of Definition 6.1.4 is independent of the parameterization. This situation is slightly more subtle for line integrals (Definition 6.2.1).

Theorem 6.3.2 (Reparameterization for line integrals)**.** *Suppose* $\mathbf{F}$ *is a vector field on the piecewise* C^1 *path* $\mathbf{r} \colon [a, b] \to \mathbb{R}^n$*, and let* $\tilde{\mathbf{r}} \colon [c, d] \to \mathbb{R}^n$ *be a reparameterization of* $\mathbf{r}$*. If* $\tilde{\mathbf{r}}$ *is orientation-preserving, then*

$$\int_{\tilde{\mathbf{r}}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s},$$

and if $\tilde{\mathbf{r}}$ *is orientation-reversing, then*

$$\int_{\tilde{\mathbf{r}}} \mathbf{F} \cdot d\mathbf{s} = - \int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s} \,.$$

Proof. As in the proof of Theorem 6.3.1, we define g as in (6.2) and set $t = g(u)$. Then

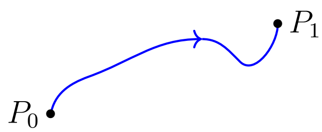
$$dt = g'(u)\,du, \qquad \tilde{\mathbf{r}}'(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{r}(g(u)) = \mathbf{r}'(g(u))g'(u).$$

Then

$$\begin{aligned}
\int_{\mathbf{r}} \mathbf{F}\cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\
&= \int_{g^{-1}(a)}^{g^{-1}(b)} \mathbf{F}(\mathbf{r}(g(u)))\cdot\mathbf{r}'(g(u))g'(u)\,du \\
&= \int_{g^{-1}(a)}^{g^{-1}(b)} \mathbf{F}(\tilde{\mathbf{r}}(u))\cdot\tilde{\mathbf{r}}'(u)\,du \\
&= \begin{cases} \int_c^d \mathbf{F}(\tilde{\mathbf{r}}(u))\cdot\tilde{\mathbf{r}}'(u)\,du & \text{if } \tilde{\mathbf{r}} \text{ is orientation-preserving,} \\ \int_d^c \mathbf{F}(\tilde{\mathbf{r}}(u))\cdot\tilde{\mathbf{r}}'(u)\,du & \text{if } \tilde{\mathbf{r}} \text{ is orientation-reversing} \end{cases} \\
&= \begin{cases} \int_{\tilde{\mathbf{r}}} \mathbf{F}\cdot d\mathbf{s} & \text{if } \tilde{\mathbf{r}} \text{ is orientation-preserving,} \\ -\int_{\tilde{\mathbf{r}}} \mathbf{F}\cdot d\mathbf{s} & \text{if } \tilde{\mathbf{r}} \text{ is orientation-reversing.} \end{cases}
\end{aligned}$$

□

Theorem 6.3.2 states that line integrals are independent of parameterization of a curve, except that reversing orientation introduces a negative sign. It thus makes sense to speak of a *oriented curve* which is, by definition, a curve C together with a choice of start point P_0 and endpoint P_1. We indicate the orientation of a curve by drawing an arrow along the curve, in the direction from P_0 to P_1:



A parameterization of such an oriented curve is a parameterization $\mathbf{r}\colon [a,b] \to \mathbb{R}^n$ such that

$$\mathbf{r}(a) = P_0 \quad \text{and} \quad \mathbf{r}(b) = P_1.$$

Then, by Theorem 6.3.2, for an oriented curve C, we can define

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{r}} \mathbf{F}\cdot d\mathbf{s}, \tag{6.3}$$

where $\mathbf{r}$ is any parameterization of C.

Exercises.

6.3.1. Complete the proof of Theorem 6.3.1 by treating the case where g is decreasing.

Further exercises are deferred until Section 6.5.

6.4 Line integrals of conservative vector fields

It turns out that computing line integrals of conservative vector fields is particularly easy.

Theorem 6.4.1 (Fundamental theorem of line integrals)**.** *Suppose* $\mathbf{F} = \nabla\varphi$ *is a conservative vector field. Then if* C *is any oriented curve that starts at* P_0 *and ends at* P_1*, we have*

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \varphi(P_1) - \varphi(P_0).$$

Proof. We give the proof for the three-dimensional setting; the proof in two dimensions is almost identical. Let

$$\mathbf{r}(t) = (x(t), y(t), z(t)), \quad a \le t \le b,$$

be a parameterization of C, with $\mathbf{r}(a) = P_0$ and $\mathbf{r}(b) = P_1$. Then,

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\
&= \int_a^b \nabla\varphi(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\
&= \int_a^b \left(\frac{\partial\varphi}{\partial x}(x(t),y(t),z(t))\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial\varphi}{\partial y}(x(t),y(t),z(t))\frac{\mathrm{d}y}{\mathrm{d}t}(t)\right. \\
&\qquad\qquad\left. + \frac{\partial\varphi}{\partial z}(x(t),y(t),z(t))\frac{\mathrm{d}z}{\mathrm{d}t}(t)\right)dt \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi(x(t),y(t),z(t))\right)dt \qquad \text{(by the chain rule)} \\
&= \varphi(\mathbf{r}(b)) - \varphi(\mathbf{r}(a)) \qquad \text{(by the fundamental theorem of calculus)} \\
&= \varphi(P_1) - \varphi(P_0).
\end{aligned}$$

□

Provided we are integrating a conservative vector field (whose potential we can find), Theorem 6.4.1 gives us an *extremely* easy way to compute line integrals. In one dimension, Theorem 6.4.1 reduces to the usual fundamental theorem of calculus.

Example 6.4.2. Recall from Example 5.2.3 that the gravitational force field $\mathbf{F}$ is conservative, with potential

$$\varphi(\mathbf{r}) = \frac{GMm}{r}, \quad \text{where } r = \|\mathbf{r}\|.$$

Thus, the work done by gravity on a particle moving along a curve C, starting at point (x_0, y_0, z_0) and ending at point (x_1, y_1, z_1) is

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \frac{GMm}{\sqrt{x_1^2+y_1^2+z_1^2}} - \frac{GMm}{\sqrt{x_0^2+y_0^2+z_0^2}}.$$

The work does not depend on the curve, only its starting point and endpoint. Furthermore, it does not depend on how long it takes the particle to travel between these two points. This agrees with our computation in Example 6.2.4. △

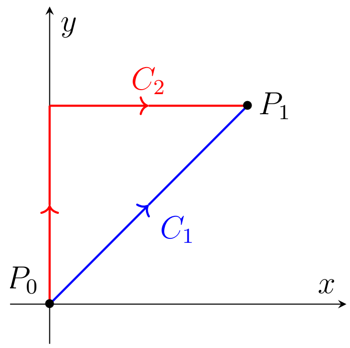
Example 6.4.3. Consider the vector field

$$\mathbf{F}(x, y) = xy\mathbf{i} + (y^2 + 1)\mathbf{j}.$$

Consider two curves, both starting at $P_0 = (0, 0)$ and ending at $P_1 = (1, 1)$.

(a) Let C_1 be the straight line from P_0 to P_1.

(b) Let C_2 be the curve, made from two straight lines, which follows the y-axis from P_0 to $(0, 1)$ and then follows the line $x = 1$ from $(1, 0)$ to P_1.



Let's compute the line integral $\int_{C_i} \mathbf{F} \cdot d\mathbf{s}$ for each curve.

(a) We parameterize C_1 by

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad t \in [0, 1].$$

Then

$$\mathbf{F}(\mathbf{r}(t)) = t^2\mathbf{i} + (t^2 + 1)\mathbf{j} \quad \text{and} \quad \mathbf{r}'(t) = \mathbf{i} + \mathbf{j}.$$

and so

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt &= \int_0^1 \left(t^2\mathbf{i} + (t^2 + 1)\mathbf{j}\right) \cdot (\mathbf{i} + \mathbf{j})\, dt \\ &= \int_0^1 (2t^2 + 1)\, dt = \left[\frac{2t^3}{3} + t\right]_{t=0}^1 = \frac{5}{3}. \end{aligned}$$

(b) We parameterize C_2 with a piecewise C^1 path. The curve C_2 is the union of the curves C_3 and C_4, where C_3 is the straight line from $(0, 0)$ to $(0, 1)$ and C_4 is the straight line from $(0, 1)$ to $(1, 1)$. We parameterize C_3 by

$$\mathbf{r}(t) = t\mathbf{j}, \quad t \in [0, 1],$$

and C_4 by

$$\mathbf{p}(t) = t\mathbf{i} + \mathbf{j}, \quad t \in [0, 1].$$

Then

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{s} = \int_{C_3} \mathbf{F} \cdot d\mathbf{s} + \int_{C_4} \mathbf{F} \cdot d\mathbf{s}$$

$$
\begin{aligned}
&= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt + \int_0^1 \mathbf{F}(\mathbf{p}(t)) \cdot \mathbf{p}'(t)\,dt \\
&= \int_0^1 (t^2+1)\mathbf{j} \cdot \mathbf{j}\,dt + \int_0^1 (t\mathbf{i} + 2\mathbf{j}) \cdot \mathbf{i}\,dt \\
&= \int_0^1 (t^2+1)\,dt + \int_0^1 t\,dt \\
&= \left[\frac{t^3}{3} + t\right]_{t=0}^1 + \left[\frac{t^2}{2}\right]_{t=0}^1 = \frac{11}{6}.
\end{aligned}
$$

We see that the integrals along C_1 and C_2 are different, even though they have the same starting point and endpoint. (See [FRYe, Example 2.4.3] for integrals along some additional curves.) Using Theorem 6.4.1, this implies that $\mathbf{F}$ is not a conservative vector field. △

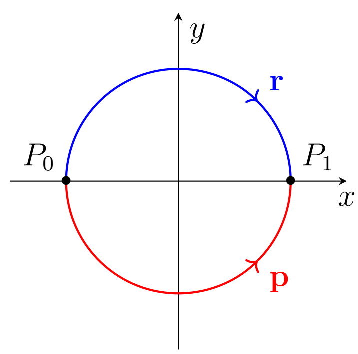
Example 6.4.4. Consider the vector field

$$\mathbf{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

We claimed in Example 5.3.6 that this vector field is *not* conservative, even though it is curl free (that is, $\nabla \times \mathbf{F} = \mathbf{0}$). One way to prove it is not conservative, is to show that the integrals along two curves with the same starting point and endpoint are different.

Let

$$
\begin{gathered}
P_0 = (-1,0), \qquad P_1 = (1,0), \\
\mathbf{r}(t) = (-\cos t, \sin t), \quad t \in [0,\pi], \\
\mathbf{p}(t) = (-\cos t, -\sin t), \quad t \in [0,\pi].
\end{gathered}
$$



Then we have

$$
\begin{aligned}
\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^\pi \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^\pi (-\sin t, -\cos t) \cdot (\sin t, \cos t)\,dt \\
&= \int_0^\pi (-\sin^2 t - \cos^2 t)\,dt \\
&= \int_0^\pi (-1)\,dt
\end{aligned}
$$

$$= -\pi$$

and

$$\begin{aligned}
\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{\pi} \mathbf{F}(\mathbf{p}(t)) \cdot \mathbf{p}'(t)\, dt \\
&= \int_0^{\pi} (\sin t, -\cos t) \cdot (\sin t, -\cos t)\, dt \\
&= \int_0^{\pi} (\sin^2 t + \cos^2 t)\, dt \\
&= \int_0^{\pi} dt \\
&= \pi.
\end{aligned}$$

Since the two integrals are different, the vector field $\mathbf{F}$ cannot be conservative. △

Remark 6.4.5. Note that the vector field $\mathbf{F}$ of Example 6.4.4 is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$, and that the paths C_1 and C_2 go on opposite sides of the origin. In a certain sense $\mathbf{F}$ is the *only* vector field defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ that has zero curl but is not conservative. More precisely, any other such vector field is a scalar multiple of $\mathbf{F}$ plus a conservative vector field. This is related to an advanced mathematical theory known as de Rham cohomology.

Exercises.

Exercises are deferred until Section 6.5.

6.5 Path independence

As we saw in Theorem 6.4.1, the line integral of a conservative vector field is *path independent*, in the sense that it depends only on the starting point and endpoint of the path. In this section, we examine this phenomenon in more detail. A good reference for the material in this section is [FRYe, §2.4.1].

To simplify our discussion, we will restrict our attention to vector fields that are defined and continuous on *all* of $\mathbb{R}^2$ or $\mathbb{R}^3$. Remember that a curve is *closed* it is starts and ends at the same point.

Theorem 6.5.1. *Suppose that* $\mathbf{F}$ *is a vector field that is defined and continuous on all of* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*. Then the following statements are equivalent:*

(a) $\mathbf{F}$ *is conservative.*

(b) *For any closed curve* C*, we have* $\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$.

(c) *The integral* $\int \mathbf{F} \cdot d\mathbf{s}$ *is path independent. More precisely, for any points* P_0, P_1*, we have* $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$ *for all curves* C_1, C_2 *that start at* P_0 *and end at* P_1*.*

Proof. (a) $\implies$ (b): Suppose (a) is true. Thus, there exists a function φ such that $\mathbf{F} = \nabla\varphi$. Then, if C is a closed curve that starts and ends at P_0, we have, by Theorem 6.4.1,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \varphi(P_0) - \varphi(P_0) = 0.$$

(b) $\implies$ (c): Suppose (b) is true. Let C_1, C_2 be paths that start at P_0 and end at P_1. Define $\tilde{C}_2$ to be the opposite of the path C_2 (that is, we follow C_2 in the opposite direction). Then let C be the path C_1 followed by the path $\tilde{C}_2$. Thus C is a closed path starting and ending at P_0. Then, using (b) and Theorem 6.3.2, we have

$$0 = \oint_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\tilde{C}_2} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$

Hence $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$.

(c) $\implies$ (a): Suppose (c) is true. We want to find a function φ such that $\mathbf{F} = \nabla\varphi$. By Theorem 6.4.1, such a function would have to satisfy

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \varphi(\mathbf{x}) - \varphi(\mathbf{0})$$

for any path C from $\mathbf{0}$ to $\mathbf{x}$. Since adding a constant to a potential yields another potential, we can assume that $\varphi(\mathbf{0}) = 0$. So let's *define* φ by

$$\varphi(\mathbf{x}) = \int_C \mathbf{F} \cdot d\mathbf{s},$$

where C is any path from $\mathbf{0}$ to $\mathbf{x}$. Since we're assuming (c) is true, this definition of $\varphi(\mathbf{x})$ does not depend on the path C we choose.

Now we want to show that $\mathbf{F} = \nabla\varphi$. Fix a point $\mathbf{x}$ and any curve $C_{\mathbf{x}}$ that starts at $\mathbf{0}$ and ends at $\mathbf{x}$. Then, for any vector $\mathbf{u}$, let $D_{\mathbf{u}}$ be the curve with parameterization

$$\mathbf{r}_{\mathbf{u}}(t) = \mathbf{x} + t\mathbf{u}, \quad 0 \le t \le 1.$$

For $s \in \mathbb{R}$, let $C_{\mathbf{x}} + D_{s\mathbf{u}}$ be the curve that first follows $C_{\mathbf{x}}$ from $\mathbf{0}$ to $\mathbf{x}$ and then follows $D_{s\mathbf{u}}$ from $\mathbf{x}$ to $\mathbf{x} + s\mathbf{u}$. Then

$$\begin{aligned}\varphi(\mathbf{x} + s\mathbf{u}) &= \int_{C_{\mathbf{x}}+D_{s\mathbf{u}}} \mathbf{F} \cdot d\mathbf{s} \\ &= \int_{C_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{s} + \int_{D_{s\mathbf{u}}} \mathbf{F} \cdot d\mathbf{s} \\ &= \int_{C_{\mathbf{x}}} \mathbf{F} \cdot d\mathbf{s} + \int_0^1 \mathbf{F}(\mathbf{x} + ts\mathbf{u}) \cdot (s\mathbf{u})\, dt,\end{aligned}$$

since $\mathbf{r}'_{s\mathbf{u}} = s\mathbf{u}$. In the second integral, make the change of variables

$$v = ts, \quad dv = s\,dt\,.$$

Then

$$\varphi(\mathbf{x} + s\mathbf{u}) = \int_{C_\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} + \int_0^s F(\mathbf{x} + v\mathbf{u}) \cdot \mathbf{u}\,dv\,.$$

The first integral is independent of s. Using the fundamental theorem of calculus for the second integral, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi(\mathbf{x} + s\mathbf{u})\Big|_{s=0} = \mathbf{F}(\mathbf{x} + s\mathbf{u}) \cdot \mathbf{u}\Big|_{s=0} = \mathbf{F}(\mathbf{x}) \cdot \mathbf{u}.$$

Applying this with $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ gives

$$\nabla\varphi(\mathbf{x}) = \left(\frac{\partial\varphi}{\partial x}(\mathbf{x}), \frac{\partial\varphi}{\partial y}(\mathbf{x}), \frac{\partial\varphi}{\partial z}(\mathbf{x})\right) = (\mathbf{F}(\mathbf{x}) \cdot \mathbf{i}, \mathbf{F}(\mathbf{x}) \cdot \mathbf{j}, \mathbf{F}(\mathbf{x}) \cdot \mathbf{k}) = \mathbf{F}(\mathbf{x}),$$

as desired. □

We can use Theorem 6.5.1 to completely characterize conservative vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$.

Theorem 6.5.2. *Let* $\mathbf{F}$ *be a vector field that is defined and* C^1 *on all of* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*. Then* $\mathbf{F}$ *is conservative if and only if* $\nabla \times \mathbf{F} = \mathbf{0}$ *(i.e.* $\mathbf{F}$ *is curl free).*

Proof. We'll give the proof for the $\mathbb{R}^3$ case. The proof for the $\mathbb{R}^2$ case is very similar and can be found in [FRYe, Th. 2.4.8]. We've already seen in Theorem 5.3.3 that every conservative vector field is curl free. It remains to show the reverse implication.

Suppose $\mathbf{F} = (F_1, F_2, F_3)$ is a vector field that is defined and C^1 on all of $\mathbb{R}^3$. Furthermore, suppose that $\nabla \times \mathbf{F} = \mathbf{0}$. Thus

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}. \tag{6.4}$$

We want to find a function φ such that $\mathbf{F} = \nabla\varphi$. So we want

$$\frac{\partial\varphi}{\partial x}(x,y,z) = F_1(x,y,z), \quad \frac{\partial\varphi}{\partial y}(x,y,z) = F_2(x,y,z), \quad \frac{\partial\varphi}{\partial z}(x,y,z) = F_3(x,y,z). \tag{6.5}$$

The function φ obeys the first equation in (6.5) if and only if there is a function $\psi(y,z)$ such that

$$\varphi(x,y,z) = \int_0^x F_1(X,y,z)\,dX + \psi(y,z).$$

This will satisfy the second equation in (6.5) if and only if

$$F_2(x,y,z) = \frac{\partial\varphi}{\partial y}(x,y,z) = \frac{\partial}{\partial y}\left(\int_0^x F_1(X,y,z)\,dX + \psi(y,z)\right)$$

$$= \int_0^x \frac{\partial F_1}{\partial y}(X, y, z)\, dX + \frac{\partial \psi}{\partial y}(y, z).$$

So we have

$$\begin{aligned}
\frac{\partial \psi}{\partial y}(y, z) &= F_2(x, y, z) - \int_0^x \frac{\partial F_1}{\partial y}(X, y, z)\, dX \\
&= F_2(x, y, z) - \int_0^x \frac{\partial F_2}{\partial x}(X, y, z)\, dX && \text{(by (6.4))} \\
&= F_2(x, y, z) - F_2(X, y, z)\Big|_{X=0}^{x} && \text{(by the fundamental theorem of calculus)} \\
&= F_2(0, y, z).
\end{aligned}$$

Thus, we can choose

$$\psi(y, z) = \int_0^y F_2(0, Y, z)\, dZ + \chi(z)$$

for some function χ depending only on $\chi(z)$.

So now we have

$$\varphi(x, y, z) = \int_0^x F_1(X, y, z)\, dX + \int_0^y F_2(0, Y, z)\, dZ + \chi(z).$$

This satisfies the last equation in (6.5) if and only if

$$\begin{aligned}
F_3(x, y, z) &= \frac{\partial \varphi}{\partial z}(x, y, z) \\
&= \frac{\partial}{\partial z}\left(\int_0^x F_1(X, y, z)\, dX + \int_0^y F_2(0, Y, z)\, dZ + \chi(z)\right) \\
&= \int_0^x \frac{\partial F_1}{\partial z}(X, y, z)\, dX + \int_0^y \frac{\partial F_2}{\partial z}(0, Y, z)\, dZ + \frac{\partial \chi}{\partial z}(z) \\
&= \int_0^x \frac{\partial F_3}{\partial x}(X, y, z)\, dX + \int_0^y \frac{\partial F_3}{\partial y}(0, Y, z)\, dZ + \frac{\partial \chi}{\partial z}(z) && \text{(by (6.4))} \\
&= F_3(X, y, z)\Big|_{X=0}^{x} + F_3(0, Y, z)\Big|_{Y=0}^{y} + \frac{\partial \chi}{\partial z}(z).
\end{aligned}$$

So we have

$$\frac{\partial \chi}{\partial z}(z) = F_3(x, y, z) - F_3(X, y, z)\Big|_{X=0}^{x} - F_3(0, Y, z)\Big|_{Y=0}^{y} = F_3(0, 0, z).$$

Thus, we can choose

$$\chi(z) = \int_0^z F_3(0, 0, Z)\, dZ + C$$

for some constant $C \in \mathbb{R}$.

Summarizing, we have found that

$$\varphi(x, y, z) = \int_0^x F_1(X, y, z)\, dX + \int_0^y F_2(0, Y, z)\, dZ + \int_0^z F_3(0, 0, Z)\, dZ + C$$

is a potential for $\mathbf{F}$. □

Later, when we learn Stokes' Theorem, we'll be able to give an alternative proof of Theorem 6.5.2. See Remark 8.4.9.

Warning 6.5.3. It is crucial in Theorem 6.5.2 that $\nabla \times \mathbf{F} = \mathbf{0}$ on *all* of $\mathbb{R}^2$ or $\mathbb{R}^3$. We saw in Examples 5.3.6 and 6.4.4 an example of a vector field that is defined and curl free on $\mathbb{R}^2 \setminus \{(0,0)\}$ but is not conservative.

Remark 6.5.4. In fact, we can generalize Theorem 6.5.1 somewhat. We can replace $\mathbb{R}^2$ or $\mathbb{R}^3$ by any region that is simply connected. Intuitively, a region is simply connected if any closed curve can be contracted to a point. The region $\mathbb{R}^2 \setminus \{(0,0)\}$ is *not* simply connected since loops around the origin *cannot* be contracted to a point.

Exercises.

Exercises from [FRYd, §2.4]: Q1–Q38.

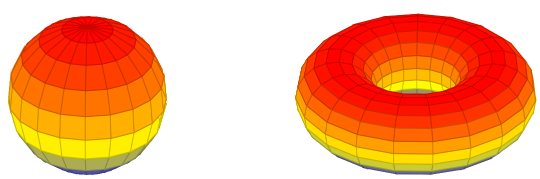
Chapter 7

Surface integrals

In this chapter we discuss integration of real-valued functions and vector fields over surfaces. A good reference for the material in this section is [FRYe, Ch. 3].

7.1 Parameterized surfaces

In Chapter 6, we saw that, in order to integrate along curves, we needed to parameterize them. The same is true for surfaces. One type of surface we've seen is the graph of a function $f(x, y)$ of two variables. However, if we only considered graphs, we would not be able to handle many important surfaces, such as spheres and tori.



Definition 7.1.1 (Parameterized surface)**.** A *parameterization of a surface* is a function

$$\Phi\colon D \to \mathbb{R}^3, \qquad D \subseteq \mathbb{R}^2.$$

The *surface* corresponding to the function Φ is its image $S = \Phi(D)$. We can write

$$\Phi(u, v) = (\phi_1(u, v), \phi_2(u, v), \phi_3(u, v)).$$

If Φ is differentiable or is of class C^1, we call S a *differentiable* or a C^1 *surface*, respectively.

We will typically require that a parameterization is injective except perhaps on a finite union of curves.

Example 7.1.2 (Graphs). If $g\colon D \to \mathbb{R}$ is a function, with $D \subseteq \mathbb{R}^2$, then

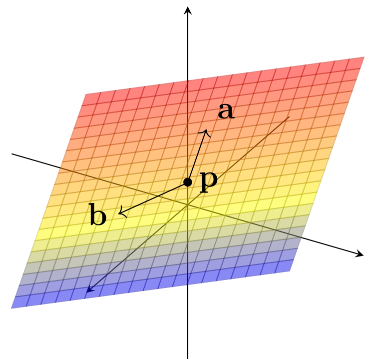
$$\Phi(u, v) = (u, v, g(u, v)), \quad (u, v) \in D.$$

is a parameterization of the graph of g. $\triangle$

Example 7.1.3 (Planes). Let P be a plane that is parallel to two vectors $\mathbf{a}$ and $\mathbf{b}$ and passes through the point $\mathbf{p}$. Then

$$\Phi(u, v) = \mathbf{p} + u\mathbf{a} + v\mathbf{b}$$

is a parameterization of P.



△

Example 7.1.4 (Sphere). Using spherical coordinates (see Section 4.5), we can parameterize the sphere of radius R centred at the origin by

$$\Phi(u, v) = (R \sin v \cos u, R \sin v \sin u, R \cos v), \quad (u, v) \in [0, 2\pi] \times [0, \pi]. \qquad \triangle$$

Example 7.1.5 (Torus). The torus obtained by revolving a circle of radius r in the yz-plane, centred at $(0, R, 0)$, about the z-axis, can be parameterized by

$$\Phi(\theta, \psi) = \big((R + r \cos\theta)\cos\psi, (R + r\cos\theta)\sin\psi, r\sin\theta\big), \quad 0 \le \theta, \psi \le 2\pi.$$

See [FRYe, Example 3.1.5] for details. △

Exercises.

Exercises from [FRYd, §3.1]: Q1–Q7.

7.2 Tangent planes

We discussed tangent planes to graphs in Section 1.4. Let's now discuss the more general situation of a tangent planes to surfaces.

Suppose we have a parameterized surface

$$\Phi = (\phi_1, \phi_2, \phi_3) \colon D \to \mathbb{R}^3.$$

Define

$$\Phi_u := \left(\frac{\partial \phi_1}{\partial u}, \frac{\partial \phi_2}{\partial u}, \frac{\partial \phi_3}{\partial u}\right), \qquad \Phi_v := \left(\frac{\partial \phi_1}{\partial v}, \frac{\partial \phi_2}{\partial v}, \frac{\partial \phi_3}{\partial v}\right).$$

Then $\Phi_u(u_0, v_0)$ and $\Phi_v(u_0, v_0)$ are tangent to the surface at $\Phi(u_0, v_0)$. Thus, if they are linearly independent, they generate the tangent plane. In other words, the tangent plane to the surface at the point $\Phi(u_0, v_0)$ is given by

$$\Phi(u_0, v_0) + s\Phi_u(u_0, v_0) + t\Phi_v(u_0, v_0), \quad s, t \in \mathbb{R}.$$

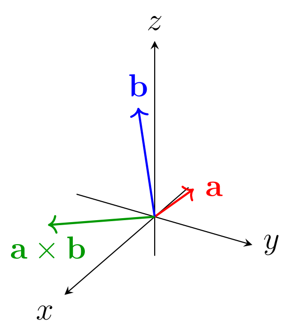
Recall that if $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} \in (b_1, b_2, b_3) \in \mathbb{R}^3$, then their *cross product*

$$\mathbf{a} \times \mathbf{b} := (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

is a vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (with direction given by the right-hand rule) and with magnitude

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.



It follows that

$$\Phi_u(u_0, v_0) \times \Phi_v(u_0, v_0)$$

is a normal vector to the surface parameterized by Φ at the point $\Phi(u_0, v_0)$, that is, it is orthogonal to the tangent plane at that point. This vector is nonzero precisely when $\Phi_u(u_0, v_0)$ and $\Phi_v(u_0, v_0)$ are linearly independent.

Definition 7.2.1 (Regular point)**.** We say that a surface S parameterized by Φ is *regular* at $\Phi(u_0, v_0) \in S$ when $\Phi_u(u_0, v_0) \times \Phi_v(u_0, v_0) \neq \mathbf{0}$.

Example 7.2.2. Let's find the tangent plane to the *helicoid* parameterized by

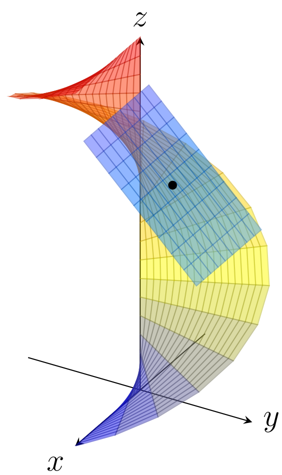
$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$$

at the point $\Phi(1, \pi)$. We have

$$\begin{aligned} \Phi_r(r,\theta) &= (\cos\theta, \sin\theta, 0), & \Phi_r(1,\pi) &= (-1, 0, 0), \\ \Phi_\theta(r,\theta) &= (-r\sin\theta, r\cos\theta, 1), & \Phi_\theta(1,\pi) &= (0, -1, 1), \end{aligned}$$

and $\Phi(1, \pi) = (-1, 0, \pi)$. Therefore, the tangent plane is given by

$$(-1, 0, \pi) + s(-1, 0, 0) + t(0, -1, 1) = (-1 - s, -t, \pi + t), \quad s, t \in \mathbb{R}.$$



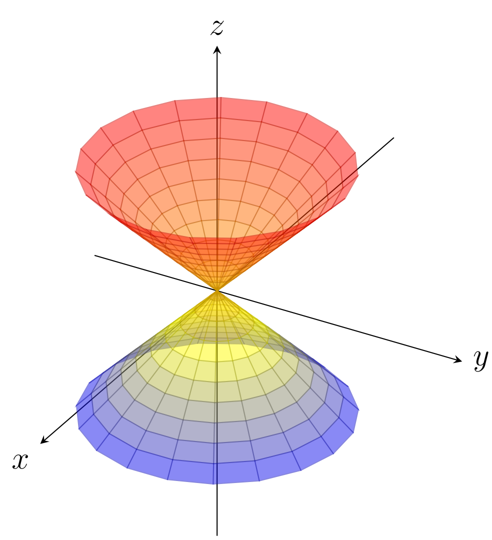
Note also that

$$\Phi_r \times \Phi_\theta = (\sin\theta, -\cos\theta, r\cos^2\theta + r\sin^2\theta) = (\sin\theta, -\cos\theta, r). \tag{7.1}$$

This vector is never equal to $\mathbf{0}$ (since $\sin\theta$ and $\cos\theta$ are never zero for the same value of θ) and so the helicoid is regular at all points. △

Example 7.2.3. Consider the cone parameterized by

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r).$$



We have

$$\Phi_r = (\cos\theta, \sin\theta, 1), \qquad \Phi_\theta = (-r\sin\theta, r\cos\theta, 0).$$

Thus

$$\Phi_r \times \Phi_\theta = (-r\cos\theta, -r\sin\theta, r\cos^2\theta + r\sin^2\theta) = (-r\cos\theta, -r\sin\theta, r).$$

Therefore the cone is regular everywhere except at the origin (when $r = 0$). △

Exercises.

Exercises from [FRYd, §3.2]: Q1–Q17.

7.3 Surface area

The first type of surface integral we consider is the computation of the surface area. Throughout the remainder of this chapter we make the following assumption.

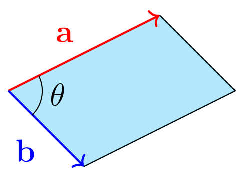
Assumption 7.3.1. We assume that all surfaces are unions of images of parameterized surfaces $\Phi_i \colon D_i \to \mathbb{R}^3$ for which:

(a) D_i is an elementary region in the plane,

(b) Φ_i is of class C^1 and one-to-one, except possibly on the boundary of D_i, and

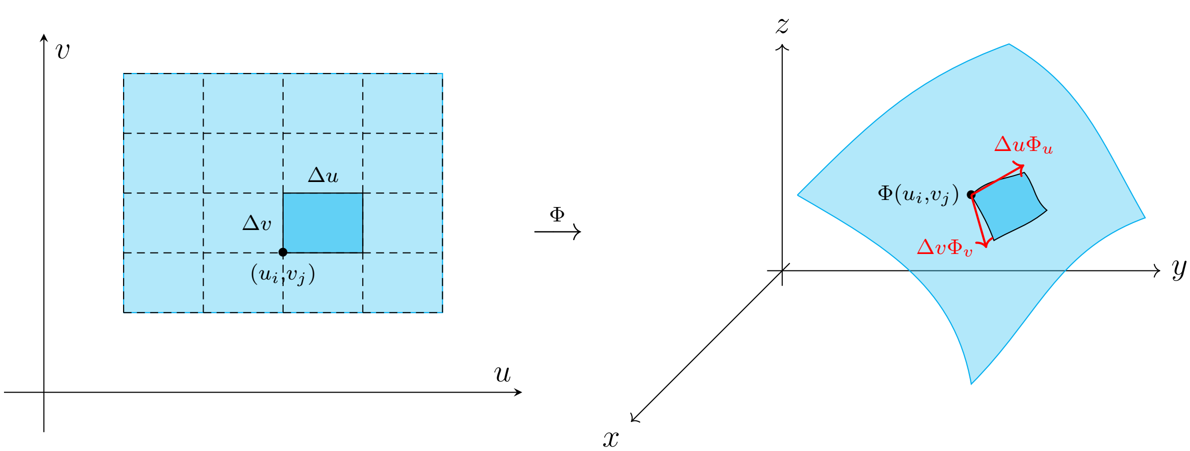
(c) the image of Φ_i is regular, except possibly at a finite number of points.

In what follows, we will use the fact $\|\mathbf{a} \times \mathbf{b}\|$ is the area of the parallelogram determined by the vectors $\mathbf{a}$ and $\mathbf{b}$.



$$\text{Area} = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta = \|\mathbf{a} \times \mathbf{b}\|. \tag{7.2}$$

Before defining general integrals of scalar functions over surfaces, we start with the definition of surface area. For simplicity, we assume that D is a rectangle. As in Section 3.3, we subdivide the rectangle D by subdividing each of its sides into n segments of equal length.



The sub-rectangle with bottom-right corner at (u_i, v_j) is sent to a piece of the surface including the point $\Phi(u_i, v_j)$. We approximate this piece of the surface by the parallelogram determined by the vectors $\Delta u \Phi_u(u_i, v_j)$ and $\Delta v \Phi_v(u_i, v_j)$. The area of this parallelogram is

$$\|\Delta u \Phi_u(u_i, v_j) \times \Delta v \Phi_v(u_i, v_j)\| = \|\Phi_u(u_i, v_j) \times \Phi_v(u_i, v_j)\| \Delta u \Delta v. \tag{7.3}$$

Adding up all the areas of these parallelograms, our approximation for the area of the surface is

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \|\Phi_u(u_i, v_j) \times \Phi_v(u_i, v_j)\| \Delta u \Delta v. \tag{7.4}$$

As $n \to \infty$, these sums should converge to the area of the surface. This leads us to the following definition.

Definition 7.3.2 (Area of a parameterized surface)**.** The *surface area* of a surface S parameterized by $\Phi\colon D \to \mathbb{R}^3$ is

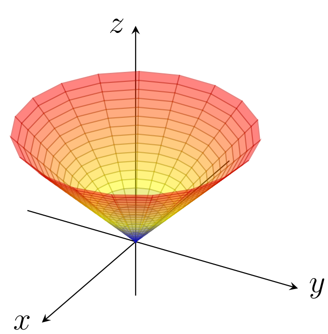
$$\operatorname{Area}(S) := \iint_D \|\Phi_u \times \Phi_v\| \, dA\,. \tag{7.5}$$

If S is a union of surfaces S_i, then its area is the sum of the areas of the S_i.

Note that our definition of surface area involves the parameterization. We will see in Theorem 7.7.2 that, in fact, it is independent of the parameterization. We saw such a phenomenon already for path integrals in Theorem 6.3.1.

Example 7.3.3 (Area of a cone)*.* Let's compute the area of the cone parameterized by

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, r), \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$



As we computed in Example 7.2.3,

$$\Phi_r \times \Phi_\theta = (-r\cos\theta, -r\sin\theta, r)$$

and $(0, 0, 0)$ is the only point where the surface is not regular. Furthermore, Φ is one-to-one on the interior of its domain $D = [0, 1] \times [0, 2\pi]$. Since D is an elementary region, Assumption 7.3.1 is satisfied. We have

$$\|\Phi_r \times \Phi_\theta\| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2}r.$$

Thus, the area of the cone is

$$\iint_D \|\Phi_r \times \Phi_\theta\| \, dr\, d\theta = \int_0^{2\pi}\int_0^1 \sqrt{2}r \, dr\, d\theta = 2\sqrt{2}\pi \left[\frac{r^2}{2}\right]_{r=0}^1 = \sqrt{2}\pi. \qquad \triangle$$

Example 7.3.4. Let's compute the surface area of the helicoid (see Example 7.2.2) parameterized by

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, \theta), \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

We have

$$\Phi_r(r,\theta) = (\cos\theta, \sin\theta, 0), \quad \Phi_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 1),$$

and so

$$\Phi_r \times \Phi_\theta = (\sin\theta, -\cos\theta, r\cos^2\theta + r\sin^2\theta) = (\sin\theta, -\cos\theta, r).$$

Since $\Phi_r \times \Phi_\theta$ is never $\mathbf{0}$ (since $\sin\theta$ and $\cos\theta$ are never zero for the same value of θ), the surface is regular at all points. The domain $[0,1] \times [0,2\pi]$ is an elementary region, and Φ is one-to-one. Thus, Assumption 7.3.1 is satisfied. We have

$$\|\Phi_r \times \Phi_\theta\| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{r^2+1}.$$

Thus, with the use of a table of integrals, we can compute that the surface area is

$$\iint\limits_D \|\Phi_r \times \Phi_\theta\|\, dr\, d\theta = \int_0^{2\pi} \int_0^1 \sqrt{r^2+1}\, dr\, d\theta = \pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right). \qquad \triangle$$

Example 7.3.5 (Graphs). Recall, from Example 7.1.2, that if $g\colon D \to \mathbb{R}$ is a function, with $D \subseteq \mathbb{R}^2$, then

$$\Phi(x,y) = (x, y, g(x,y)), \quad (x,y) \in D,$$

is a parameterization of the graph of f. Then we have

$$\Phi_x(x,y) = \left(1, 0, \frac{\partial g}{\partial x}(x,y)\right) \quad \text{and} \quad \Phi_y(x,y) = \left(0, 1, \frac{\partial g}{\partial y}(x,y)\right),$$

and so

$$\Phi_x \times \Phi_y = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right),$$

which is never zero. Thus, provided D is an elementary region and f is C^1, Assumption 7.3.1 is satisfied. Therefore, the surface area of the graph is

$$\text{Area(graph of } g) = \iint\limits_D \left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\right) dx\, dy\,. \tag{7.6}$$

$\triangle$

Exercises.

Exercises from [FRYd, §3.3]: Q1–Q3, Q5–Q10, Q14

7.4 Integrals of scalar functions over surfaces

In Section 7.3 we learned how to compute the area of a surface. Now suppose that the surface is a thin sheet whose density at point (x, y, z) is given by a continuous function $f(x, y, z)$. Now we want to compute the *mass* of the surface. In this case, we should multiply the area of the parallelogram given in (7.3) by the density $f(\Phi(u_i^*, v_j^*))$ for some point (u_i^*, v_j^*) in the sub-rectangle with bottom-right corner at (u_i, v_j). This gives an approximation of the mass of the image of this rectangle under the parameterization $\Phi\colon D \to \mathbb{R}^3$. Adding up these approximations, our approximation for the mass of the surface is

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(\Phi(u_i^*, v_j^*))\|\Phi_u(u_i, v_j) \times \Phi_v(u_i, v_j)\|\Delta u \Delta v.$$

Compare to (7.4). As $n \to \infty$, these sums should converge to the mass of the surface. This leads us to the following definition, which applies more generally (i.e. not only when the function $f(x, y, z)$ represents a density).

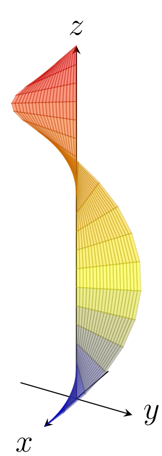
Definition 7.4.1 (Integral of a scalar function over a surface)**.** If $f(x, y, z)$ is a real-valued continuous function defined on a parameterized surface $\Phi\colon D \to \mathbb{R}^3$ satisfying Assumption 7.3.1, we define the *integral of f over* Φ to be

$$\iint_\Phi f(x, y, z)\, dS = \iint_\Phi f\, dS = \iint_D f(\Phi(u, v))\|\Phi_u \times \Phi_v\|\, dA\,. \tag{7.7}$$

Note that when $f(x, y, z) = 1$, (7.7) reduces to the formula (7.5) for surface area, as expected.

Example 7.4.2. Consider the helicoid S from Example 7.2.2 parameterized by

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, \theta), \quad (r, \theta) \in D = [0, 1] \times [0, 2\pi].$$



Let's compute

$$\iint_\Phi f\, dA, \quad \text{where } f(x, y, z) = \sqrt{x^2 + y^2}.$$

We computed in (7.1) that

$$\Phi_r \times \Phi_\theta = (\sin\theta, -\cos\theta, r).$$

Thus

$$\|\Phi_r \times \Phi_\theta\| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1+r^2}.$$

We also have

$$f(\Phi(u,v)) = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2} = r \quad \text{since } r \geq 0.$$

Therefore,

$$\begin{aligned}
\iint_\Phi \sqrt{x^2+y^2}\, dA &= \iint_D r\sqrt{1+r^2}\, dA \\
&= \int_0^1 \int_0^{2\pi} r\sqrt{1+r^2}\, d\theta\, dr \\
&= \int_0^1 2\pi r\sqrt{1+r^2}\, dr\,.
\end{aligned}$$

We now make the substitution

$$u = 1 + r^2, \quad du = 2r\, dr\,.$$

Then

$$\iint_\Phi \sqrt{x^2+y^2}\, dA = \int_1^2 \pi\sqrt{u}\, du = \left[\frac{2\pi}{3}u^{3/2}\right]_{u=1}^2 = \frac{2\pi}{3}(2\sqrt{2}-1). \qquad \triangle$$

Example 7.4.3 (Graphs). In Example 7.3.5, we developed a formula for the surface area of a graph (which is a special type of surface). Repeating the argument there, but for the integral of a function $f(x,y,z)$ over the graph, we can give a formula for the integral of a scalar function over a graph. Suppose

$$g\colon D \to \mathbb{R}, \quad D \subseteq \mathbb{R}^2$$

is a C^1 function and D is an elementary region. Consider the graph of g, with parameterization

$$\Phi(x,y) = (x,y,g(x,y)), \quad (x,y) \in D.$$

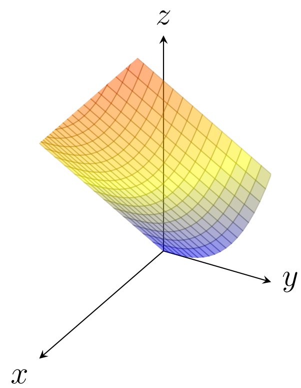
Then the integral of a continuous function f over Φ is

$$\iint_D f(x,y,g(x,y)) \left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\right) dx\, dy\,. \tag{7.8}$$

Note that, when $f(x,y,z) = 1$, (7.8) reduces to (7.6). $\triangle$

Example 7.4.4. Let's integrate y^3 over the surface defined by

$$z = x + y^3, \quad 0 \le x \le 3,\ 0 \le y \le 1.$$



Note that this surface is the graph of the function

$$g(x, y) = x + y^3.$$

Thus, (7.8) gives

$$\begin{aligned}
\iint\limits_{\Phi} y^3 \, dS &= \iint\limits_{D} y^3 \left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} \right) dA \\
&= \int_0^1 \int_0^3 y^3 \sqrt{1 + 9y^4 + 1} \, dx \, dy \\
&= \int_0^1 3y^3 \sqrt{1 + 9y^4 + 1} \, dy \\
&= \int_2^{11} \frac{1}{12} \sqrt{u} \, du && (u = 9y^4 + 2,\ du = 36y^3 \, dy) \\
&= \left[\frac{1}{18} u^{3/2} \right]_{u=2}^{u=11} = \frac{1}{18} \left(11^{3/2} - 2^{3/2} \right). && \triangle
\end{aligned}$$

Note that Definition 7.4.1 depends on a choice of parameterization of the surface. Later, in Section 7.7, we will see that, in fact, the integral does not depend on the parameterization.

Exercises.

Exercises from [FRYd, §3.3]: Q11, Q15, Q20, Q25, Q26, Q32, Q34

7.5 Centre of mass and moment of inertia

In this section we briefly discuss some applications of integration in higher dimensions in physics.

Definition 7.5.1 (Centre of mass)**.** Suppose an object occupies a volume $V \subseteq \mathbb{R}^3$, and its density at point (x, y, z) is given by $\rho(x, y, z)$. The *centre of mass* of the object is (x_0, y_0, z_0), where

$$x_0 = \frac{1}{M} \iiint_V x\rho(x, y, z)\, dV,$$

$$y_0 = \frac{1}{M} \iiint_V y\rho(x, y, z)\, dV,$$

$$z_0 = \frac{1}{M} \iiint_V z\rho(x, y, z)\, dV,$$

and

$$M = \iiint_V \rho(x, y, z)\, dV$$

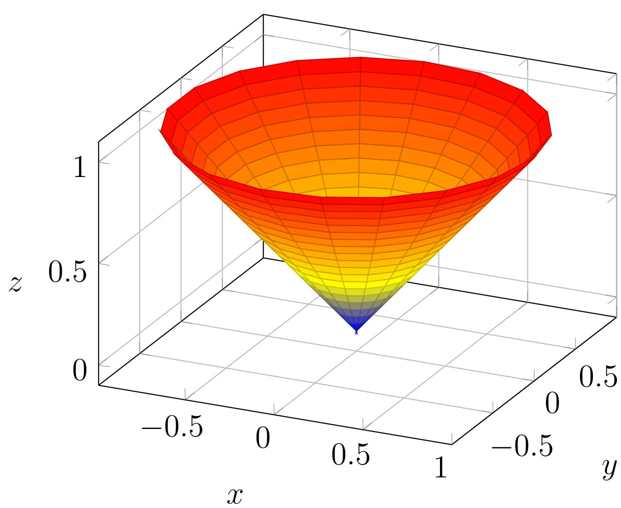
is the total mass of the object. If the object is one-dimensional or two-dimensional, we replace the above integrals by path or surface integrals, respectively. If the density of the object is not given, we assume it has constant density $\rho(x, y, z) = 1$.

Example 7.5.2. Consider the solid cone

$$V = \{(x, y, z) : \sqrt{x^2 + y^2} \leq z \leq 1\}$$

and its surface

$$S = \{(x, y, z) : z = \sqrt{x^2 + y^2} \leq 1\}.$$



If both have constant density 1, which is more stable? More precisely, which has a lower centre of mass?

Let's first consider the solid cone. Using cylindrical coordinates, we compute that its mass is

$$M_V = \iiint\limits_V dV = \int_0^1 \int_0^z \int_0^{2\pi} r\, d\theta\, dr\, dz = 2\pi \int_0^1 \left[\frac{r^2}{2}\right]_{r=0}^z dz$$
$$= 2\pi \int_0^1 \frac{z^2}{2}\, dz = 2\pi \left[\frac{z^3}{6}\right]_{z=0}^1 = \frac{\pi}{3}.$$

Next, we compute

$$\iiint\limits_V z\, dV = \int_0^1 \int_0^z \int_0^{2\pi} zr\, d\theta\, dr\, dz = 2\pi \int_0^1 \left[\frac{zr^2}{2}\right]_{r=0}^z dz$$
$$= 2\pi \int_0^1 \frac{z^3}{2}\, dz = 2\pi \left[\frac{z^4}{8}\right]_{z=0}^1 = \frac{\pi}{4}.$$

Thus

$$z_0 = \frac{\frac{\pi}{4}}{\frac{\pi}{3}} = \frac{3}{4}.$$

By symmetry, we have $x_0 = y_0 = 0$. Thus, the centre of mass of the solid cone is $\left(0, 0, \frac{3}{4}\right)$.

Now let's consider the empty cone. Our computation of the surface area of a cone in Example 7.3.3 gives

$$M_S = \sqrt{2}\pi.$$

As in Example 7.3.3, we parameterize the cone by

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, r), \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi,$$

and we have

$$\|\Phi_r \times \Phi_\theta\| = \sqrt{2}r.$$

Thus,

$$\iint\limits_S z\, dA = \int_0^1 \int_0^{2\pi} r\sqrt{2}r\, d\theta\, dr = 2\pi \int_0^1 \sqrt{2}r^2\, dr = \left[\frac{2\sqrt{2}\pi}{3} r^3\right]_{r=0}^1 = \frac{2\sqrt{2}\pi}{3}.$$

Therefore,

$$z_0 = \frac{\frac{2\sqrt{2}\pi}{3}}{\sqrt{2}\pi} = \frac{2}{3}.$$

Again, by symmetry, we have $x_0 = y_0 = 0$. Thus, the centre of mass is $\left(0, 0, \frac{2}{3}\right)$. So the empty cone has a lower centre of mass than the solid cone. $\triangle$

For the next definition, we consider rotating some object about an axis. The axis is a line in $\mathbb{R}^3$. The moment of inertia about this axis measures the torque needed per unit of angular acceleration. It is the rotational analogue of usual inertia.

Definition 7.5.3 (Moment of inertia)**.** Suppose an object occupies a volume $V \subseteq \mathbb{R}^3$, and its density at point (x, y, z) is given by $\rho(x, y, z)$. Fix an axis, and let $d(x, y, z)$ denote the distance from the point (x, y, z) to the axis. Then the *moment of inertia* of the object about the axis is

$$\iiint_V \rho(x,y,z)d(x,y,z)^2\, dV\,.$$

If the object is one-dimensional or two-dimensional, we replace the above integrals by path or surface integrals, respectively. If the density of the object is not given, we assume it has constant density $\rho(x, y, z) = 1$.

Example 7.5.4. Consider the cube

$$V = [0,1]^3$$

with density

$$\rho(x,y,z) = x.$$

Its moment of inertia about the x-axis is

$$\begin{aligned}
\iiint_V \rho(x,y,z)d(x,y,z)^2\, dV &= \int_0^1 \int_0^1 \int_0^1 x(y^2+z^2)\, dx\, dy\, dz \\
&= \int_0^1 \int_0^1 \left[\frac{x^2}{2}(y^2+z^2)\right]_{x=0}^1 dy\, dz \\
&= \frac{1}{2}\int_0^1 \int_0^1 (y^2+z^2)\, dy\, dz \\
&= \frac{1}{2}\int_0^1 \left[\frac{y^3}{3}+z^2y\right]_{y=0}^1 dz \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{3}+z^2\right) dz \\
&= \frac{1}{2}\left[\frac{z}{3}+\frac{z^3}{3}\right]_{z=0}^1 \\
&= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{1}{3}.
\end{aligned}$$

△

Example 7.5.5. Consider the half-sphere

$$S = \{(x,y,z) : x^2+y^2+z^2 = 1,\ z \geq 0\}$$

with density

$$\rho(x,y,z) = z.$$

Let's find its moment of inertia about the z-axis. As in Example 7.7.1, we parameterize this surface by

$$\Phi(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi), \quad \theta \in [0,2\pi],\ \varphi \in [0, \tfrac{\pi}{2}],$$

and we have

$$\|\Phi_\theta \times \Phi_\varphi\| = \|(-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi)\|$$

$$= \sqrt{\cos^2\theta \sin^4\varphi + \sin^2\theta\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$
$$= \sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = \sqrt{\sin^2\varphi} = \sin\varphi,$$

since $\sin\varphi \geq 0$ for $\varphi \in [0, \frac{\pi}{2}]$. Then the moment of inertia about the z-axis is

$$\iint_S z(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^{\pi/2} \cos\varphi\left(\cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi\right)\sin\varphi\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/2}\cos\varphi\sin^3\varphi\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\left[\frac{1}{4}\sin^4\varphi\right]_{\varphi=0}^{\pi/2} d\theta = \int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\pi}{2}. \qquad \triangle$$

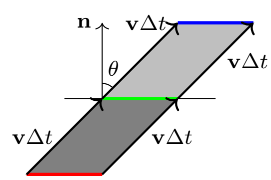
Exercises.

Exercises from [FRYd, §3.3]: Q4, Q13, Q21, Q22, Q27,

7.6 Surface integrals of vector fields

In Section 7.4, we developed the notion of the integral of a *scalar* function over a surface. We now want to define the integral of a *vector* field over a surface. This is analogous to the two types of integrals along curves. Definition 6.2.1 was motivated by the concept of *work*; see Definition 6.2.3. Our definition of surface integrals of vector fields will be inspired by the physical notion of *flux*. A good reference for the material in this section is [FRYe, §3.4].

If our vector field $\mathbf{v}(x, y, z)$ gives the velocity of a fluid, $\rho(x, y, z)$ denotes the density of the fluid, and we put an imaginary surface in the fluid, we can ask: "What is the rate at which the fluid is crossing the surface"? Let $\mathbf{n}$ be the unit normal vector to a small piece of the surface, with area ΔS. We draw a cross section as follows:



The green line is a side view of the piece ΔS of the surface. In a small time interval Δt,

- the red line moves to the green line and
- the green line moves to the blue line, so that
- the fluid filling the dark grey region below the green line crosses through ΔS and moves to the light grey region above the green line.

If we denote by θ the angle between $\mathbf{n}$ and $\mathbf{v}$, the dark grey region has base ΔS and height $\|\mathbf{v}\Delta t\|\cos\theta$. Thus, its volume is

$$\|\mathbf{v}(x,y,z)\Delta t\|\cos\theta\Delta S = \mathbf{v}(x,y,z)\cdot\mathbf{n}(x,y,z)\Delta t\Delta S,$$

because $\mathbf{n}(x,y,z)$ has length one. The mass of the fluid that crosses ΔS during the time interval Δt is then

$$\rho(x,y,z)\mathbf{v}(x,y,z)\cdot\mathbf{n}(x,y,z)\Delta t\Delta S$$

and the rate at which fluid is crossing through ΔS is

$$\rho(x,y,z)\mathbf{v}(x,y,z)\cdot\mathbf{n}(x,y,z)\Delta S.$$

Adding up these amounts over all pieces ΔS of the surface and then taking the limit as the size of the pieces ΔS and the time interval Δt goes to zero, we find that the rate at which fluid is crossing through the surface S is the *flux integral*

$$\iint_S \rho(x,y,z)\mathbf{v}(x,y,z)\cdot\mathbf{n}(x,y,z)\,dS\,.$$

Now, comparing to (7.7), we see that to compute this integral, we should parameterize the surface by a function $\Phi\colon D\to\mathbb{R}^3$. Then dS is replaced by $\|\Phi_u\times\Phi_v\|\,dA$, and we integrate over D. Recall that the point $\Phi(u_0,v_0)\in S$ is *regular* if $\Phi_u(u_0,v_0)\times\Phi_v(u_0,v_0)\neq\mathbf{0}$. Since Φ_u and Φ_v are tangent to the surface, their cross product $\Phi_u\times\Phi_v$ is normal to the surface. Therefore, informally speaking, we have

$$\mathbf{n}\,dS = \Phi_u\times\Phi_v.$$

This motivates the following definition (letting $\mathbf{F}=\rho\mathbf{v}$).

Definition 7.6.1 (Integral of a vector field over a surface)**.** Let $\mathbf{F}$ be a vector field defined on the image of a parameterized surface Φ. The *surface integral* of $\mathbf{F}$ over Φ is

$$\iint_\Phi \mathbf{F}\cdot d\mathbf{S} := \iint_D \mathbf{F}\cdot(\Phi_u\times\Phi_v)\,dA\,.$$

Note that Definition 7.6.1 depends on a choice of parameterization of the surface. Later, in Section 7.7, we will see that, up to a sign, the integral does not depend on the parameterization.

Example 7.6.2. Consider the cylinder S parameterized by

$$\Phi\colon[0,2\pi]\times[0,1]\to\mathbb{R}^3,\quad \Phi(u,v)=(\cos u,\sin u,v).$$

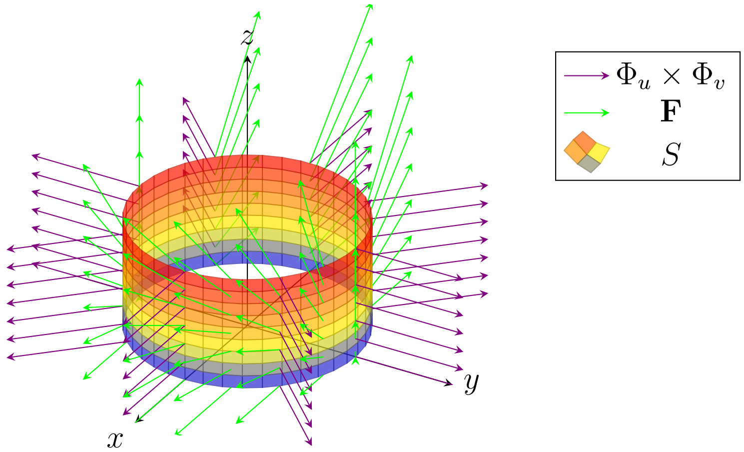
and the vector field

$$\mathbf{F}(x,y,z)=(x,0,z).$$

We have

$$\Phi_u\times\Phi_v = (-\sin u,\cos u,0)\times(0,0,1)=(\cos u,\sin u,0).$$

Drawing this surface and vector field, we have the following:



We compute

$$\mathbf{F}(\Phi(u,v)) \cdot (\Phi_u \times \Phi_v) = (\cos u, 0, v) \cdot (\cos u, \sin u, 0) = \cos^2 u = \tfrac{1}{2}(1 + \cos(2u)).$$

Thus,

$$\iint_\Phi \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} \frac{1}{2}(1 + \cos(2u))\, du\, dv = \frac{1}{2} \int_0^{2\pi} (1 + \cos(2u))\, du$$

$$= \frac{1}{2} \left[u + \frac{1}{2} \sin(2u) \right]_{u=0}^{2\pi} = \pi. \quad \triangle$$

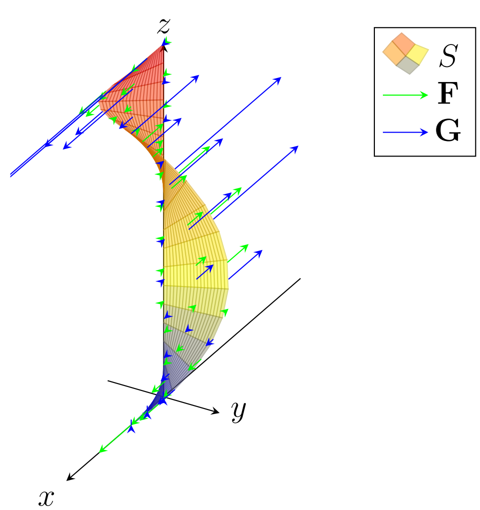
Example 7.6.3. Recall the helicoid S of Examples 7.2.2 and 7.4.2, with parameterization

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, \theta), \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

Let's integrate the following vector fields over Φ:

$$\mathbf{F}(x, y, z) = (x, 0, 0), \qquad \mathbf{G}(x, y, z) = (xz, 0, 0).$$

So we have the following picture:



First we compute

$$\Phi_r \times \Phi_\theta = (\cos\theta, \sin\theta, 0) \times (-r\sin\theta, r\cos\theta, 1) \\ = (\sin\theta, -\cos\theta, r\cos^2\theta + r\sin^2\theta) = (\sin\theta, -\cos\theta, r).$$

Since

$$\mathbf{F}(\Phi(r,\theta)) \cdot (\Phi_r \times \Phi_\theta) = (r\cos\theta, 0, 0) \cdot (\sin\theta, -\cos\theta r) = r\cos\theta\sin\theta = \tfrac{r}{2}\sin(2\theta),$$

we have

$$\iint_\Phi \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} \frac{r}{2}\sin(2\theta)\, d\theta\, dr = \int_0^1 \left[-\frac{r}{4}\cos(2\theta)\right]_{\theta=0}^{2\pi} dr = \int_0^1 0\, dr = 0.$$

On the other hand,

$$\mathbf{G}(\Phi(r,\theta)) \cdot (\Phi_r \times \Phi_\theta) = (r\theta\cos\theta, 0, 0) \cdot (\sin\theta, -\cos\theta, r) = r\theta\cos\theta\sin\theta = \tfrac{r\theta}{2}\sin(2\theta),$$

and so

$$\iint_\Phi \mathbf{G} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 \frac{r\theta}{2}\sin(2\theta)\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r^2\theta}{4}\sin(2\theta)\right]_{r=0}^1 d\theta = \frac{1}{4}\int_0^{2\pi} \theta\sin(2\theta)\, d\theta\,.$$

We calculate the last integral using integration by parts with

$$\begin{aligned} u &= \theta, & dv &= \sin(2\theta)\, dv, \\ du &= d\theta, & v &= -\tfrac{1}{2}\cos(2\theta). \end{aligned}$$

So,

$$\int \theta\sin(2\theta)\, d\theta = -\tfrac{1}{2}\theta\cos(2\theta) - \int -\tfrac{1}{2}\cos(2\theta)\, d\theta = -\tfrac{1}{2}\theta\cos(2\theta) + \tfrac{1}{4}\sin(2\theta).$$

Therefore

$$\iint_\Phi \mathbf{G} \cdot d\mathbf{S} = \frac{1}{4}\left[-\tfrac{1}{2}\theta\cos(2\theta) + \tfrac{1}{4}\sin(2\theta)\right]_{\theta=0}^{2\pi} = \frac{1}{4}\left((-\pi + 0) - (0+0)\right) = -\frac{\pi}{4}.$$ △

Example 7.6.4 (Graphs). Recall, from Example 7.3.5, that if $g\colon D \to \mathbb{R}$ is a function, with $D \subseteq \mathbb{R}^2$, then

$$\Phi(x,y) = (x, y, g(x,y)), \quad (x,y) \in D,$$

is a parameterization of the graph of g and we have

$$\Phi_x \times \Phi_y = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right).$$

Therefore, the integral of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ over the graph is given by

$$\iint_\Phi \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-F_1\frac{\partial g}{\partial x} - F_2\frac{\partial g}{\partial y} + F_3\right) dx\, dy\,.$$ △

Exercises.

Exercises from [FRYd, §3.3]: Q16, Q23, Q24, Q28–Q31, Q33, Q35–Q37

7.7 Reparameterization and orientation

In Section 6.3 we discussed the effect of reparameterization on line integrals. In particular, we saw that a line integral is independent of parameterization provided that we fix an orientation of the curve over which we are integrating. This allowed us to define, in (6.3), an integral over an oriented curve. We would now like to do the same for surface integrals.

Suppose

$$\Phi\colon D \to \mathbb{R}^3, \quad \Psi\colon E \to \mathbb{R}^3$$

are two parameterizations of the same surface S. In other words, Φ, Ψ are one-to-one C^1 functions with $\Phi(D) = \Phi(E) = S$. Choose a point $(x, y, z) \in S$. Then there exist unique

$$(u, v) \in D \text{ such that } \Phi(u, v) = (x, y, z) \quad \text{and} \quad (s, t) \in E \text{ such that } \Psi(s, t) = (x, y, z).$$

We say that Φ and Ψ have the *same orientation* at (x, y, z) if

$$\Phi_u \times \Phi_v \quad \text{and} \quad \Psi_s \times \Psi_t$$

are positive scalar multiples of each other. We say they have they have *opposite orientation* at (x, y, z) if they are negative scalar multiples of each other. (Since $\Phi_u \times \Phi_v$ and $\Psi_s \times \Psi_t$ are both normal to S, these are the only possibilities, provided both vectors are nonzero.) We say that Φ and Ψ have the same (respectively, opposite) orientation if they have the same (respectively, opposite) orientation at all points of S.

Example 7.7.1. Consider the half-sphere

$$\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$$

with parameterizations

$$\begin{aligned} \Phi(\theta, \varphi) &= (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi), && \theta \in [0, 2\pi],\ \varphi \in [0, \tfrac{\pi}{2}], \\ \Psi(u, v) &= (u, v, \sqrt{1 - u^2 - v^2}), && (u, v) \in \{(x, y) : x^2 + y^2 \leq 1\}. \end{aligned}$$

(Note that Φ is not actually one-to-one as given. To fix this, we could modify the domain to be $\big([0, 2\pi] \times (0, \frac{\pi}{2}] \cup \{(0, 0)\}$.) We compute

$$\begin{aligned} \Phi_\theta \times \Phi_\varphi &= (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0) \times (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi) \\ &= (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi), \\ \Psi_u \times \Psi_v &= \left(1, 0, \frac{-2u}{2\sqrt{1 - u^2 - v^2}}\right) \times \left(0, 1, \frac{-2v}{2\sqrt{1 - u^2 - v^2}}\right) \end{aligned}$$

$$= \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1 \right).$$

(Note that $\Phi_u \times \Phi_v$ is not defined on the boundary $u^2+v^2=1$ of its domain.) For $\varphi \in (0, \frac{\pi}{2}]$, we have $-\sin\varphi\cos\varphi < 0$. So the z-coordinate of $\Phi_\theta \times \Phi_\varphi$ is negative, while the z-coordinate of $\Psi_u \times \Psi_v$ is positive. Thus, Φ and Ψ have opposite orientations.

To further illustrate this, let's consider a specific point, say

$$\Psi\left(\tfrac{1}{\sqrt{2}}, 0\right) = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right) = \Phi(0, \tfrac{\pi}{4}).$$

At this point, the normal vectors coming from the two parameterizations are

$$\Phi_\theta \times \Phi_\varphi\left(0, \tfrac{\pi}{4}\right) = \left(-\tfrac{1}{2}, 0, -\tfrac{1}{2}\right). \quad \text{and} \quad \Psi_u \times \Psi_v\left(\tfrac{1}{\sqrt{2}}, 0\right) = (1, 0, 1).$$

So $\Psi_u \times \Psi_v = -2\Phi_\theta \times \Phi_\varphi$ at this point. Hence the orientations are opposite. △

We are now ready to state our main theorem of this section.

Theorem 7.7.2. *Suppose Φ, Ψ are two one-to-one C^1 parameterizations of the same surface S. Let $\mathbf{F}$ be a vector field on S and let f be a real-valued function on S.*

(a) *If Φ and Ψ have the same orientation then*

$$\iint\limits_{\Phi} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{\Psi} \mathbf{F} \cdot d\mathbf{S}.$$

(b) *If Φ and Ψ have opposite orientations, then*

$$\iint\limits_{\Phi} \mathbf{F} \cdot d\mathbf{S} = -\iint\limits_{\Psi} \mathbf{F} \cdot d\mathbf{S}.$$

(c) *We have*

$$\iint\limits_{\Phi} f\, dS = \iint\limits_{\Psi} f\, dS.$$

Proof. The proof of this theorem is similar to that of Theorem 6.3.1. It involves using a change of variables. We omit the details. □

Theorem 7.7.2 tells us that integrals of scalar functions over surfaces are independent of parameterization, while integrals of vector fields over surfaces depend only on the orientation of the parameterization. This is analogous to Theorems 6.3.1 and 6.3.2, which state that integrals of scalar functions along paths are independent of parameterization, while integrals of vector fields along paths depend only on the orientation of the parameterization.

In light of Theorem 7.7.2, if f is a real-valued function defined on a surface S, we can define

$$\iint\limits_{S} f\, dS := \iint\limits_{\Phi} f\, dS,$$

where Φ is *any* regular parameterization of S.

For vector fields, we need to specify an orientation. An *oriented surface* is a two-sided surface with one side specified as the *outside* or *positive side*. We call the other side the *inside* or *negative side*. We say that $\Phi(u,v)$ is a parameterization of the oriented surface S if $\Phi(u,v)$ is a parameterization of S and $\Phi_u \times \Phi_v$ points out of the positive side of S for all (u,v) in the domain of Φ. Then, for a vector field $\mathbf{F}$ defined on an *oriented* surface S, we can define

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} := \iint\limits_\Phi \mathbf{F} \cdot d\mathbf{S},$$

where Φ is any regular parameterization of the oriented surface S.

Example 7.7.3. Suppose we choose to orient the half-sphere S of Example 7.7.1 upward. That is, we our normal vectors to have positive z-component. Consider the vector field $\mathbf{F}(x,y,z) = (0,0,1)$. As we noted in Example 7.7.1, the parameterization Ψ gives normal vectors with positive z-components. So Ψ is a parameterization of the oriented surface S. Therefore,

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_\Psi \mathbf{F} \cdot d\mathbf{S} = \iint\limits_\Psi (0,0,1) \cdot (\Psi_u \times \Psi_v)\, dS = \iint\limits_D 1\, du\, dv = \operatorname{Area}(D) = \pi,$$

where $D = \{(x,y) : x^2 + y^2 \le 1\}$ is the domain of Ψ. On the other hand

$$\begin{aligned}\iint\limits_\Phi \mathbf{F} \cdot d\mathbf{S} = \iint\limits_\Phi (0,0,1) \cdot (\Phi_\theta \times \Phi_\varphi)\, dS &= \int_0^{\pi/2} \int_0^{2\pi} (-\sin\varphi \cos\varphi)\, d\theta\, d\varphi \\ &= -2\pi \int_0^{\pi/2} \sin\varphi \cos\varphi\, d\varphi = -2\pi \left[\frac{1}{2}\sin^2\varphi\right]_{\varphi=0}^{\pi/2} = -\pi,\end{aligned}$$

which is $-\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$ since the orientation of Φ is opposite that of S. △

If a vector field $\mathbf{F}$ represents the flow of a fluid, the sign of a surface integral indicates the flow relative to the orientation of the surface. In other words, if $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$ is positive, then there is a net flow across S in the direction of the orientation of S. On the other hand, if $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$ is negative, then the net flow is in the opposite direction.

Note that some surfaces *cannot* be oriented. In other words, there are surfaces where it is not possible to choose one side. The most famous example is the Möbius strip, which has only one side.

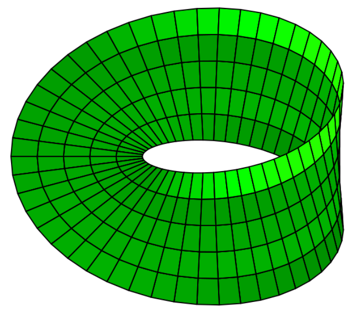


Image source: https://commons.wikimedia.org/wiki/File:Green_Coloured_Mobius_Strip.png

Integrating a vector field over the Möbius strip is problematic! For further discussion on orientation see [FRYe, §3.5].

Chapter 8

Integral theorems

In this chapter we discuss three theorems: the divergence theorem, Green's theorem, and Stokes' theorem. They are all generalization of the fundamental theorem of calculus. Recall that the fundamental theorem of calculus states that

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}t}(t)\, dt = f(b) - f(a).$$

Thus, it relates the integral of the derivative of a function f over an interval $[a, b]$ to the values of that function on the boundary $\{a, b\}$ of that interval. The divergence theorem, Green's theorem, and Stokes' theorem will also have this form. However, we will now work in higher dimensions. For the divergence theorem, the integral on the left-hand side is over a three-dimensional volume, and the right-hand side is an integral over the boundary of the volume, which is a surface. For Green's theorem and Stokes' theorem, the integral on the left-hand side is over a surface, and the right-hand side is an integral over the boundary of the surface, which is a curve. A good reference for the material in this section is [FRYe, Ch. 4].

8.1 Gradient, divergence, and curl

We begin with a discussion of gradient, divergence, and curl. We have already seen gradient and curl, but divergence is new. The best way to remember these operators uses the *nabla* symbol ∇. This is a differential operator defined by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Throughout this section, $\mathbf{F}$ and $\mathbf{G}$ denote three-dimensional vector fields and f, g, h denote scalar-valued functions. we denote the components of a vector field $\mathbf{F}$ by F_1, F_2, F_3 (and similarly for $\mathbf{G}$, etc.).

Definition 8.1.1 (Gradient, divergence, curl, Laplacian)**.** (a) The *gradient* of a scalar-valued function $f(x, y, z)$ is the vector field

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

(b) The *divergence* of a vector field $\mathbf{F}(x, y, z)$ is the scalar-valued function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

(c) The *curl* of a vector field $\mathbf{F}(x, y, z)$ is the vector field

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

(d) The *Laplacian* of a scalar-valued function $f(x, y, z)$ is the scalar-valued function

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

8.1.1 Vector identities

When working with the above operators, it is useful to have some identities at our disposal. Some of these identities involve the expression $(\mathbf{G} \cdot \nabla)\mathbf{F}$ for vector fields $\mathbf{F}, \mathbf{G}$. This is defined to be

$$(\mathbf{G} \cdot \nabla)\mathbf{F} := G_1 \frac{\partial \mathbf{F}}{\partial x} + G_2 \frac{\partial \mathbf{F}}{\partial y} + G_3 \frac{\partial \mathbf{F}}{\partial z}.$$

The proofs of all of these identities involve using rules you've learned for differentiation. So, for each proposition, we will only prove a small selection of the identities, leaving the proof of the others as an exercise. The proofs can also be found in [FRYe, §4.1.1].

Proposition 8.1.2 (Gradient identities)**.**

(a) $\nabla(f + g) = \nabla f + \nabla g$.

(b) $\nabla(cf) = c\nabla f$ *for all* $c \in \mathbb{R}$.

(c) $\nabla(fg) = (\nabla f)g + f(\nabla g)$.

(d) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ *at points* $\mathbf{x}$ *where* $g(\mathbf{x}) \neq 0$.

(e) $\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) - (\nabla \times \mathbf{F}) \times \mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G}$.

Proof. To prove (c), we compute

$$\begin{aligned}
\nabla(fg) &= \mathbf{i}\frac{\partial fg}{\partial x} + \mathbf{j}\frac{\partial fg}{\partial y} + \mathbf{k}\frac{\partial fg}{\partial z} \\
&= \mathbf{i}\left(\frac{\partial f}{\partial x} g + f\frac{\partial g}{\partial x}\right) + \mathbf{j}\left(\frac{\partial f}{\partial y} g + f\frac{\partial g}{\partial y}\right) + \mathbf{k}\left(\frac{\partial f}{\partial z} g + f\frac{\partial g}{\partial z}\right) \qquad \text{(chain rule)} \\
&= (\nabla f)g + f(\nabla g).
\end{aligned}$$

Since $\nabla c = \mathbf{0}$ for a constant $c \in \mathbb{R}$, we have

$$\nabla(cf) = (\nabla c)f + c(\nabla f) = c\nabla f,$$

proving (b). □

Proposition 8.1.3 (Divergence identities)**.**

(a) $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$.

(b) $\nabla \cdot (c\mathbf{F}) = c\nabla \cdot \mathbf{F}$ *for all* $c \in \mathbb{R}$.

(c) $\nabla \cdot (f\mathbf{F}) = (\nabla f) \cdot \mathbf{F} + f\nabla \cdot \mathbf{F}$.

(d) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$.

Proof. To prove (c), we compute

$$\begin{aligned}
\nabla \cdot (f\mathbf{F}) &= \frac{\partial (fF_1)}{\partial x} + \frac{\partial (fF_2)}{\partial y} + \frac{\partial (fF_3)}{\partial z} \\
&= \frac{\partial f}{\partial x}F_1 + f\frac{\partial F_1}{\partial x} + \frac{\partial f}{\partial y}F_2 + f\frac{\partial F_2}{\partial y} + \frac{\partial f}{\partial z}F_3 + f\frac{\partial F_3}{\partial z} \qquad \text{(product rule)} \\
&= (\nabla f) \cdot \mathbf{F} + f\nabla \cdot \mathbf{F}.
\end{aligned}$$

Taking $f(x, y, z) = c$ for some constant $c \in \mathbb{R}$ and using the fact that $\nabla c = \mathbf{0}$ then gives (b). □

Proposition 8.1.4 (Curl identities)**.**

(a) $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$.

(b) $\nabla \times (c\mathbf{F}) = c\nabla \times \mathbf{F}$ *for all* $c \in \mathbb{R}$.

(c) $\nabla \times (f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f\nabla \times \mathbf{F}$.

(d) $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$.

Proof. To prove (c), we use the chain rule to compute

$$\begin{aligned}
\frac{\partial (fF_3)}{\partial y} - \frac{\partial (fF_2)}{\partial z} &= \frac{\partial f}{\partial y}F_3 + f\frac{\partial F_3}{\partial y} - \frac{\partial f}{\partial z}F_2 - f\frac{\partial F_2}{\partial z} \\
\frac{\partial (fF_1)}{\partial z} - \frac{\partial (fF_3)}{\partial x} &= \frac{\partial f}{\partial z}F_1 + f\frac{\partial F_1}{\partial z} - \frac{\partial f}{\partial x}F_3 - f\frac{\partial F_3}{\partial x} \\
\frac{\partial (fF_2)}{\partial x} - \frac{\partial (fF_1)}{\partial y} &= \frac{\partial f}{\partial x}F_2 + f\frac{\partial F_2}{\partial x} - \frac{\partial f}{\partial y}F_1 - f\frac{\partial F_1}{\partial y}.
\end{aligned}$$

Thus

$$\begin{aligned}
\nabla \times (f\mathbf{F}) &= \left(\frac{\partial f}{\partial y}F_3 + f\frac{\partial F_3}{\partial y} - \frac{\partial f}{\partial z}F_2 - f\frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z}F_1 + f\frac{\partial F_1}{\partial z} - \frac{\partial f}{\partial x}F_3 - f\frac{\partial F_3}{\partial x}\right)\mathbf{j} \\
&\qquad + \left(\frac{\partial f}{\partial x}F_2 + f\frac{\partial F_2}{\partial x} - \frac{\partial f}{\partial y}F_1 - f\frac{\partial F_1}{\partial y}\right)\mathbf{k} \\
&= \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2\right)\mathbf{i} + \left(\frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right)\mathbf{k} \\
&\qquad + f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + f\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + f\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \\
&= (\nabla f) \times \mathbf{F} + f\nabla \times \mathbf{F}.
\end{aligned}$$

Taking $f(x, y, z) = c$ for some constant $c \in \mathbb{R}$ then gives (b). □

Proposition 8.1.5 (Laplacian identities)**.**

(a) $\nabla^2(f+g) = \nabla^2 f + \nabla^2 g$.

(b) $\nabla^2(cf) = c\nabla^2 f$ *for all* $c \in \mathbb{R}$.

(c) $\nabla^2(fg) = f\nabla^2 g + 2(\nabla f)\cdot(\nabla g) + g\nabla^2 f$.

Proof. To prove (c), we compute

$$\begin{aligned}
\nabla^2(fg) &= \nabla \cdot \nabla(fg) \\
&= \nabla \cdot \big((\nabla f)g + f(\nabla g)\big) && \text{(by Proposition 8.1.2(c))} \\
&= \nabla \cdot \big((\nabla f)g\big) + \nabla \cdot \big(f(\nabla g)\big) && \text{(by Proposition 8.1.3(a))} \\
&= (\nabla^2 f)g + (\nabla f)\cdot(\nabla g) + (\nabla f)\cdot(\nabla g) + f\nabla^2 g, && \text{(by Proposition 8.1.3(c))}
\end{aligned}$$

as desired. □

Proposition 8.1.6 (Degree two identities)**.** *Suppose* $\mathbf{F}$ *and* f, g, h *are* C^2.

(a) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

(b) $\nabla \times (\nabla f) = \mathbf{0}$.

(c) $\nabla \cdot \big(f(\nabla g \times \nabla h)\big) = \nabla f \cdot (\nabla g \times \nabla h)$.

(d) $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

Proof. We proved (b) in Theorem 5.3.3(b). □

Example 8.1.7. Let $\mathbf{r}(x,y,z) = x^2\mathbf{i} + y\mathbf{j} + z^3\mathbf{k}$, and let $\psi(x,y,z)$ be any real-valued function. Let's prove that

$$\nabla \cdot (\mathbf{r} \times \nabla\psi) = 0.$$

Using Proposition 8.1.3(d), we have

$$\nabla \cdot (\mathbf{r} \times \nabla\psi) = (\nabla \times \mathbf{r}) \cdot \nabla\psi - \mathbf{r} \cdot (\nabla \times (\nabla\psi)) = (\nabla \times \mathbf{r}) \cdot \nabla\psi,$$

where we used Proposition 8.1.6(b) to conclude that $\nabla \times (\nabla\psi) = \mathbf{0}$. Next, we compute

$$\nabla \times \mathbf{r} = \left(\frac{\partial(z^3)}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(x^2)}{\partial z} - \frac{\partial(z^3)}{\partial x}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial(x^2)}{\partial y}\right)\mathbf{k} = \mathbf{0}.$$

Thus, $\nabla \cdot (\mathbf{r} \times \nabla\psi) = 0$, as desired. In fact, this is true for *any* curl free $\mathbf{r}$. △

8.1.2 Physical interpretations

It is helpful to have an intuitive interpretation of gradient, divergence, and curl. Such intuition helps us understand some of the theorems and applications to come. We already saw, in Proposition 1.8.6, that the gradient $\nabla f(\mathbf{x})$ at a point $\mathbf{x}$ points in the direction in which f increases most rapidly at $\mathbf{x}$. Then Proposition 1.8.4 tells us that its magnitude is the rate of change in this direction.

Next we discuss divergence. Fix a point (x_0, y_0, z_0). Then, for $\varepsilon > 0$, let

$$S_\varepsilon = \{(x, y, z) : (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = \varepsilon^2\}$$

be the sphere of radius ε centred at (x_0, y_0, z_0). Then one can show that, for a C^1 vector field $\mathbf{F} \colon \mathbb{R}^3 \to \mathbb{R}^3$, we have

$$\nabla \cdot \mathbf{F}(x_0, y_0, z_0) = \lim_{\varepsilon \to 0} \frac{1}{\mathrm{Vol}(S_\epsilon)} \iint\limits_{S_\varepsilon} \mathbf{F} \cdot d\mathbf{S},$$

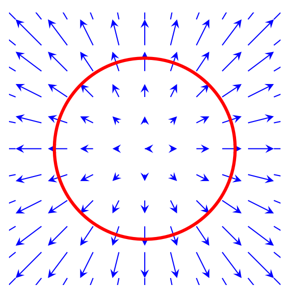
where $\mathrm{Vol}(S_\epsilon) = \frac{4}{3}\pi\varepsilon^3$ is the volume of S_ε. (For a proof of this fact, see [FRYe, Lem. 4.1.20].) If we interpret $\mathbf{F}$ as the velocity of some fluid, then the flux integral $\iint\limits_{S_\varepsilon} \mathbf{F} \cdot d\mathbf{S}$ represents the total flow across the surface. Thus

$$\nabla \cdot \mathbf{F}(\mathbf{x})$$

is the rate at which the fluid is exiting an infinitesimal sphere, centered at $\mathbf{x}$, per unit time, per unit volume. So it is a measure of the strength of the "source" at $\mathbf{x}$.

Example 8.1.8. Consider the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$



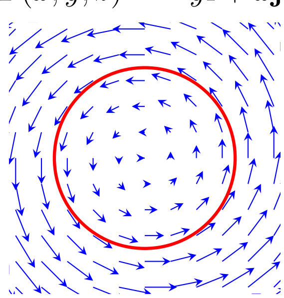
If this vector represents the flow of fluid, then fluid is being created and pushed out through any sphere centred at the origin. The divergence is

$$\nabla \cdot \mathbf{F} = 3 > 0,$$

consistent with our intuitive interpretation. △

Example 8.1.9. Consider the vector field

$$\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}.$$



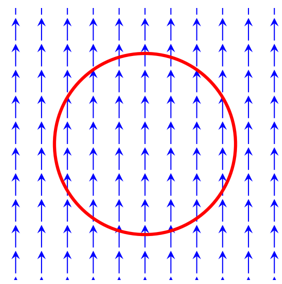
In this case, the fluid is going in circles. No fluid actually exits the sphere. The divergence is

$$\nabla \cdot \mathbf{F} = 0,$$

consistent with our interpretation. △

Example 8.1.10. Consider the vector field

$$\mathbf{F}(x, y, z) = \mathbf{j}.$$

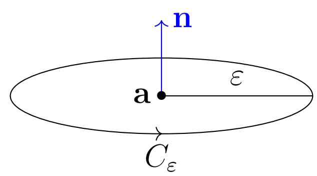


In this case, the fluid is moving uniformly upwards. Fluid enters the sphere at the bottom and leaves through the top at exactly the same rate. So the net rate at which fluid crosses the sphere is zero. The divergence is

$$\nabla \cdot \mathbf{F} = 0,$$

consistent with our interpretation. △

Finally, we turn our attention to curl. Fix $\varepsilon > 0$ and $\mathbf{a}, \mathbf{n} \in \mathbb{R}^3$. Let C_ε be the circle centred at $\mathbf{a}$, with radius ε, and in the plane through $\mathbf{a}$ orthogonal to $\mathbf{n}$. We orient C_ε using the right-hand rule. If the thumb of your right hand points in the direction of $\mathbf{n}$ and you curl your fingers, they point in the orientation of C_ε.



Then one can show that

$$\big(\nabla \times \mathbf{F}(\mathbf{a})\big) \cdot \mathbf{n} = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \oint_{C_\varepsilon} \mathbf{F} \cdot d\mathbf{s},$$

where $\pi\varepsilon^2$ is the area of the disk with boundary C_ε. (For a proof of this fact, see [FRYe, Lem. 4.1.25].) Now suppose that $\mathbf{F}$ represents the velocity of a fluid and you place an infinitesimal paddlewheel at $\mathbf{a}$ with its axle in direction $\mathbf{n}$. If the paddlewheel is rotating at a rate of Ω radians per minute, then the speed of the paddles is $\Omega\varepsilon$. (Remember that radians are defined as arc length divided by radius). This must be equal to the average value of speed of the paddles in the direction of rotation. Since the circumference of C_ε is $2\pi\varepsilon$, we have

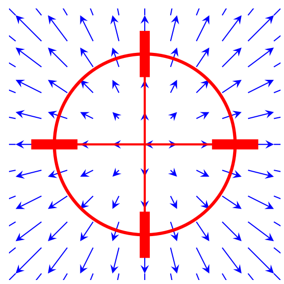
$$\Omega\varepsilon = \frac{1}{2\pi\varepsilon} \oint_{C_\varepsilon} \mathbf{F} \cdot d\mathbf{s} \implies \Omega = \frac{1}{2\pi} \oint_{C_\varepsilon} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{2}\big(\nabla \times \mathbf{F}(\mathbf{a})\big) \cdot \mathbf{n}.$$

Thus, our infinitesimal paddlewheel rotates at $\frac{1}{2}\big(\nabla \times \mathbf{F}(\mathbf{a})\big) \cdot \mathbf{n}$ radians per unit time. In particular, to maximize the rotation, we should orient the paddlewheel so that $\mathbf{n}$ is parallel to $\nabla \times \mathbf{F}(\mathbf{a})$.

Let's return to the vector fields of Examples 8.1.8, 8.1.9, and 8.1.10 and consider our interpretation of their curls.

Example 8.1.11. Consider the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$



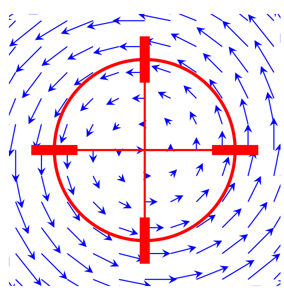
If this vector represents the flow of fluid, then fluid is moving parallel to the paddles, so the paddlewheel should not rotate at all. Indeed, we have

$$\nabla \times \mathbf{F}(\mathbf{0}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{bmatrix} = \mathbf{0} \implies \big(\nabla \times \mathbf{F}(\mathbf{0})\big) \cdot \mathbf{k} = \mathbf{0},$$

as expected. △

Example 8.1.12. Consider the vector field

$$\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}.$$



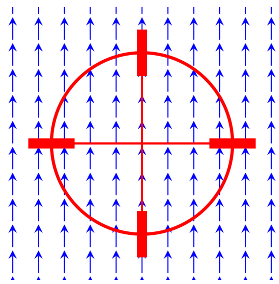
In this case, the fluid is rotating counterclockwise. So the paddlewheel should also rotate counterclockwise. Hence it should have positive angular velocity. Indeed, we have

$$\nabla \times \mathbf{F}(\mathbf{0}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{bmatrix} = 2\mathbf{k} \implies \big(\nabla \times \mathbf{F}(\mathbf{0})\big) \cdot \mathbf{k} = 2\mathbf{k} \cdot \mathbf{k} = 2 > 0,$$

as expected. △

Example 8.1.13. Consider the vector field

$$\mathbf{F}(x, y, z) = \mathbf{j}.$$



In this case, the fluid is pushing the right paddle counterclockwise and the left paddle clockwise, with the same force. So the paddle should not rotate. Indeed, we have

$$\nabla \times \mathbf{F}(\mathbf{0}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 0 & 0 \end{bmatrix} = \mathbf{0} \implies \big(\nabla \times \mathbf{F}(\mathbf{0})\big) \cdot \mathbf{k} = \mathbf{0},$$

as expected. △

Exercises.

Exercises from [FRYd, §4.1]: Q1–Q13.

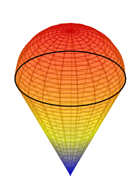
8.2 Divergence theorem

The divergence theorem will relate a volume integral over a solid V to a flux integral over the surface of V. We first start with some definitions about the type of surfaces we consider.

Definition 8.2.1. (a) A surface is *smooth* if it has a C^1 parameterization $\Phi(u, v)$ such that all points are regular, i.e. $\Phi_u \times \Phi_v \neq \mathbf{0}$ everywhere.

(b) A surface is *piecewise smooth* if it is a finite union of smooth surfaces that meet along curves.

Example 8.2.2. The ice-cream cone shape



is piecewise smooth, but not smooth, since it has a "sharp curve" where the half-sphere (the ice cream) meets the cone. △

Recall from Section 1.1 that ∂A denotes the boundary of a set A. Thus, if V is a bounded solid, then ∂V denotes the surface of V.

Theorem 8.2.3 (Divergence theorem)**.** *Suppose that V is a bounded solid with piecewise smooth boundary ∂V, and let $\mathbf{F}$ be a C^1 vector field on V. We give ∂V the outward orientation (i.e. we choose normal vectors pointing away from V). Then*

$$\iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_{V} \nabla \cdot \mathbf{F}\, dV .$$

Proof. For a proof of this theorem, see [FRYe, Th. 4.2.2]. Given the physical interpretation of divergence we discussed in Section 8.1.2, the divergence theorem should not be surprising. The right-hand integral

$$\iiint_{V} \nabla \cdot \mathbf{F}\, dV$$

adds up the divergence over all points of the solid V. For any points in the interior of V, flow out of a small sphere is cancelled by the flow into neighbouring spheres. Thus, what we are left with is the net flow out of the boundary, which is the left-hand integral

$$\iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} . \qquad \square$$

The divergence theorem is sometimes called *Gauss' theorem* or *Gauss' divergence theorem.* Like the fundamental theorem of calculus, the divergence theorem states that the integral of a derivative of a function (here the "derivative" is the divergence) over some region is equal to the integral of the function over the boundary of that region. When using the divergence theorem, we will always choose the outward orientation for the surface of a solid region.

Example 8.2.4. Consider the ball

$$V = \{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$$

and the vector field

$$\mathbf{F}(x, y, z) = (x, y, 0).$$

Let's compute both sides of the divergence theorem and verify that they are equal. The boundary ∂V of V is the unit sphere, which we can parameterize by

$$\Phi(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi).$$

As we computed in Example 7.7.1, we have

$$\Phi_\theta \times \Phi_\varphi = (-\cos\theta \sin^2\varphi, -\sin\theta \sin^2\varphi, -\sin\varphi \cos\varphi).$$

Also as we noted in Example 7.7.1, this is the inward orientation, so we will need to add a negative sign, since the divergence theorem assumes the outward orientation. (Alternatively, we could interchange θ and φ, since $\Phi_\theta \times \Phi_\varphi = -\Phi_\varphi \times \Phi_\theta$.) We have

$$\mathbf{F}(\Phi(\theta, \varphi)) \cdot (\Phi_\theta \times \Phi_\varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, 0) \cdot (-\cos\theta \sin^2\varphi, -\sin\theta \sin^2\varphi, -\sin\varphi \cos\varphi)$$

$$
\begin{aligned}
&= -\cos^2\theta\sin^3\varphi - \sin^2\theta\sin^3\varphi \\
&= -\sin^3\varphi \\
&= (\cos^2\varphi - 1)\sin\varphi.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\iint\limits_{\partial V} \mathbf{F}\cdot d\mathbf{S} &= -\iint\limits_{\Phi} \mathbf{F}\cdot d\mathbf{S} && \text{(because of the reversed orientation)} \\
&= -\int_0^{\pi}\int_0^{2\pi} (\cos^2\varphi - 1)\sin\varphi\, d\theta\, d\varphi \\
&= -2\pi\int_0^{\pi} (\cos^2\varphi - 1)\sin\varphi\, d\varphi \\
&= 2\pi\int_1^{-1} (u^2 - 1)\, du && (u = \cos\varphi,\ du = -\sin\varphi\, d\varphi) \\
&= 2\pi\left[\frac{u^3}{3} - u\right]_{u=1}^{-1} \\
&= 2\pi\left(-\tfrac{2}{3} + 2\right) = \tfrac{8\pi}{3}.
\end{aligned}
$$

On the other hand,

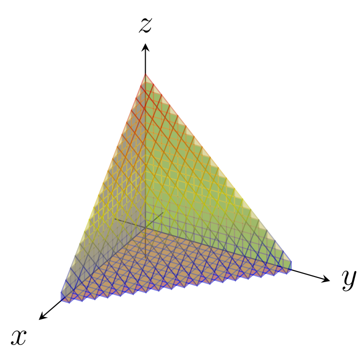
$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}0 = 2,$$

and so

$$\iiint\limits_V \nabla\cdot\mathbf{F}\, dV = \iiint\limits_V 2\, dV = 2\,\mathrm{Vol}(V) = 2\cdot\frac{4\pi}{3} = \frac{8\pi}{3}. \qquad \triangle$$

Example 8.2.5. Consider the solid

$$V = \{(x, y, z) : x, y, z \geq 0,\ x + y + z \leq 1\}$$



and the vector field

$$\mathbf{F}(x, y, z) = (xy, yz, xz).$$

The boundary ∂V has 4 components:

$$\Phi_1(u, v) = (u, v, 0), \qquad u, v \geq 0,\ u + v \leq 1,$$

$$\begin{aligned}
\Phi_2(u,v) &= (u,0,v), && u,v \geq 0,\ u+v \leq 1,\\
\Phi_3(u,v) &= (0,u,v), && u,v \geq 0,\ u+v \leq 1,\\
\Phi_4(u,v) &= (u,v,1-u-v), && u,v \geq 0,\ u+v \leq 1.
\end{aligned}$$

These components have normal vectors:

$$\begin{aligned}
\Phi_{1u} \times \Phi_{1v} &= (1,0,0)\times(0,1,0) = (0,0,1), && \text{(negative orientation)}\\
\Phi_{2u} \times \Phi_{2v} &= (1,0,0)\times(0,0,1) = (0,-1,0), && \text{(positive orientation)}\\
\Phi_{3u} \times \Phi_{3v} &= (0,1,0)\times(0,0,1) = (1,0,0), && \text{(negative orientation)}\\
\Phi_{4u} \times \Phi_{4v} &= (1,0,-1)\times(0,1,-1) = (1,1,1). && \text{(positive orientation)}
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{F}(\Phi_1(u,v))\cdot\big(\Phi_{1u}\times\Phi_{1v}\big) &= (uv,0,0)\cdot(0,0,1) = 0,\\
\mathbf{F}(\Phi_2(u,v))\cdot\big(\Phi_{2u}\times\Phi_{2v}\big) &= (0,0,uv)\cdot(0,-1,0) = 0,\\
\mathbf{F}(\Phi_3(u,v))\cdot\big(\Phi_{3u}\times\Phi_{3v}\big) &= (0,uv,0)\cdot(1,0,0) = 0,\\
\mathbf{F}(\Phi_4(u,v))\cdot\big(\Phi_{4u}\times\Phi_{4v}\big) &= (uv,v(1-u-v),u(1-u-v))\cdot(1,1,1) = u+v-uv-u^2-v^2.
\end{aligned}$$

and so

$$\begin{aligned}
\iint_{\partial V} \mathbf{F}\cdot d\mathbf{S} &= -\iint_{\Phi_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{\Phi_2} \mathbf{F}\cdot d\mathbf{S} - \iint_{\Phi_3} \mathbf{F}\cdot d\mathbf{S} + \iint_{\Phi_4} \mathbf{F}\cdot d\mathbf{S}\,.\\
&= \iint_{\Phi_4} \mathbf{F}\cdot d\mathbf{S}\\
&= \int_0^1 \int_0^{1-u} (u+v-uv-u^2-v^2)\,dv\,du\\
&= \int_0^1 \left(u(1-u) + \frac{(1-u)^2}{2} - \frac{u(1-u)^2}{2} - u^2(1-u) - \frac{(1-u)^3}{3}\right) du\\
&= \frac{1}{6}\int_0^1 (5u^3 - 9u^2 + 1)\,du\\
&= \frac{1}{6}\left(\frac{5}{4} - \frac{9}{3} + 1\right) = \frac{1}{6}\cdot\frac{3}{4} = \frac{1}{8}.
\end{aligned}$$

On the other hand

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = x+y+z,$$

and so

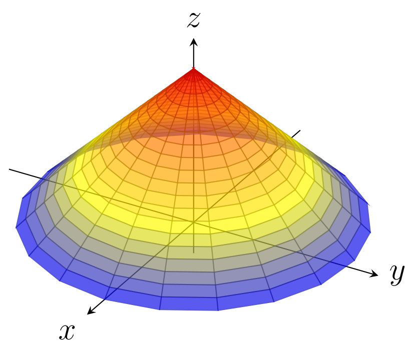
$$\begin{aligned}
\iiint_V \nabla\cdot\mathbf{F}\,dx\,dy\,dz &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\,dz\,dy\,dx\\
&= \int_0^1 \int_0^{1-x} \left((x+y)(1-x-y) + \frac{(1-x-y)^2}{2}\right) dy\,dx
\end{aligned}$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} \left(1 - x^2 - 2xy - y^2\right) dy\, dx$$
$$= \frac{1}{2}\int_0^1 \left((1-x^2)(1-x) - x(1-x)^2 - \frac{(1-x)^3}{3}\right) dx$$
$$= \frac{1}{6}\int_0^1 \left(x^3 - 3x + 2\right) dx$$
$$= \frac{1}{6}\left(\frac{1}{4} - \frac{3}{2} + 2\right) = \frac{1}{6}\cdot\frac{3}{4} = \frac{1}{8}. \qquad \triangle$$

The divergence theorem can also be used to simplify some surface integrals, as the following example illustrates.

Example 8.2.6. Consider the cone

$$C = \{(x,y,z) : x^2 + y^2 = (1-z)^2\}$$



oriented upwards and the vector field

$$\mathbf{F}(x,y,z) = (x^2, -xy, 1 - xz).$$

Note that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}(1 - xz) = 2x - x - x = 0.$$

Therefore, by the divergence theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$$

for any piecewise smooth *closed* surface S (i.e. any surface S that is the boundary of a bounded solid). Let's close off the cone by adding the bottom

$$B = \{(x,y,0) : x^2 + y^2 \le 1\}$$

oriented downwards. Then $V = C \cup B$ is a closed surface, and so

$$\iint_C \mathbf{F} \cdot d\mathbf{S} + \iint_B \mathbf{F} \cdot d\mathbf{S} = 0 \implies \iint_C \mathbf{F} \cdot d\mathbf{S} = -\iint_B \mathbf{F} \cdot d\mathbf{S}\,.$$

So we have turned our integral over C into an integral over B, which is easier to compute! We parameterize B by

$$\Phi(u,v) = (u,v,0), \quad (u,v) \in D := \{(x,y) : x^2 + y^2 \le 1\}.$$

We have

$$\Phi_u \times \Phi_v = (0,0,1),$$

which is opposite to the orientation of B. Since

$$\mathbf{F}(\Phi(u,v)) \cdot (\Phi_u \times \Phi_v) = (u^2, -uv, 1) \cdot (0,0,1) = 1,$$

we have

$$\iint_C \mathbf{F} \cdot d\mathbf{S} = -\iint_B \mathbf{F} \cdot d\mathbf{S} = \iint_\Phi \mathbf{F} \cdot d\mathbf{S} = \iint_D 1\, du\, dv = \text{Area}(D) = \pi. \qquad \triangle$$

Warning 8.2.7. In the divergence theorem, it is crucial that $\mathbf{F}$ is C^1 at *every* point of V. For example, the divergence theorem fails for

$$\mathbf{F} = \frac{\mathbf{r}}{\|\mathbf{r}\|^3}, \quad V = \{(x,y,z) : x^2 + y^2 + z^2 \le 1\},$$

where $\mathbf{r}(x,y,z) = (x,y,z)$. Note that $\mathbf{F}$ is not defined at the origin, which is contained in V. See [FRYe, Example 4.2.7] for details.

Exercises.

Exercises from [FRYd, §4.2]: Q1–Q34.

8.3 Green's theorem

Our next integral theorem is Green's theorem, which relates the integral of a derivative of a vector-valued function over a region in the plane to an integral of the function over the boundary curve. We begin by discussing a few properties of curves, some of which we have seen before. A good reference for the material in this section is [FRYe, §4.3].

Definition 8.3.1. Let $\mathbf{r}(t)$, $a \le t \le b$, be a parameterization of a curve C.

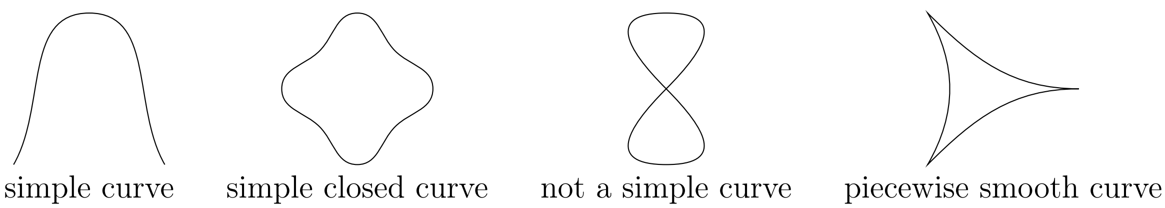
(a) The curve C is *closed* if $\mathbf{r}(a) = \mathbf{r}(b)$.

(b) The curve C is *simple* if it does not cross itself. More precisely, C is simple if, for any $a \le t_1, t_2 \le b$ satisfying $\mathbf{r}(t_1) = \mathbf{r}(t_2)$, we have

$$t_1 = t_2 \quad \text{or} \quad (t_1 = a,\ t_2 = b) \quad \text{or} \quad (t_1 = b,\ t_1 = a).$$

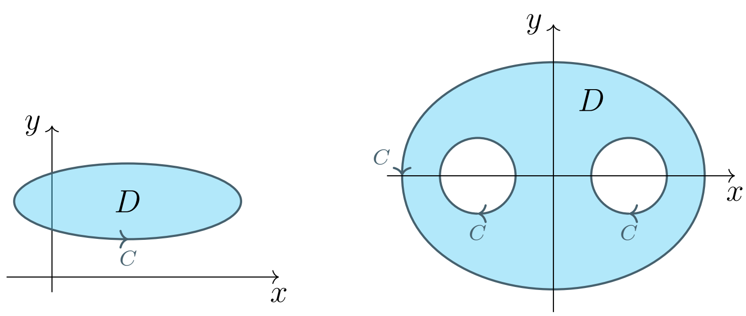
(c) The curve C is *piecewise smooth* if it has a parameterization $\mathbf{r}(t)$ that
  (i) is continuous and which
  (ii) is differentiable except possibly at finitely many points with
  (iii) the derivative being continuous and nonzero except possibly at finitely many points.

Example 8.3.2. Here are some examples of curves:



△

Theorem 8.3.3 (Green's theorem)**.** *Suppose D is a bounded region in the xy-plane and that the boundary, C, of D consists of a finite number of piecewise smooth, simple closed curves, each oriented so that if you walk along in the direction of the orientation, then D is on your left.*



Furthermore, suppose that $\mathbf{F} = (F_1, F_2)$ is a C^1 vector field on D. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint\limits_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy\,.$$

Remarks 8.3.4. (a) Remember that $\oint_C$ is just an alternate notation for $\int_C$ that is sometimes used when C is a closed curve (or a union of closed curves).

(b) Recall that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is the *scalar curl* of $\mathbf{F}$. It is the z-component of the curl of $\mathbf{F}$ if we extend $\mathbf{F}$ to be a vector field in three dimensions with zero z-component. See (5.3).

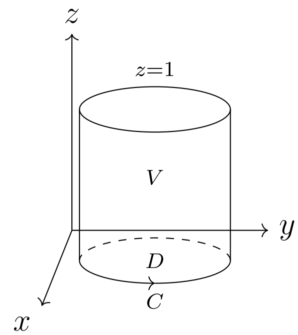
(c) Sometimes we write

$$\oint_C \big(F_1(x, y)\, dx + F_2(x, y)\, dy \big) \quad \text{for} \quad \oint_C \mathbf{F} \cdot d\mathbf{s}\,.$$

Proof Theorem 8.3.3. We will use the divergence theorem (Theorem 8.2.3) to prove Green's theorem. Define

$$V = \{(x, y, z) : (x, y) \in D,\ 0 \le z \le 1\},$$

$$\mathbf{G}(x, y, z) = F_2(x, y)\mathbf{i} - F_1(x, y)\mathbf{j}.$$



By the divergence theorem, we have

$$\iint_{\partial V} \mathbf{G} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{G}\, dV\,. \tag{8.1}$$

The right-hand side of (8.1) is

$$\begin{aligned}\iiint_V \nabla \cdot \mathbf{G}\, dV &= \iiint_V \left(\frac{\partial F_2}{\partial x}(x, y) - \frac{\partial F_1}{\partial y}(x, y)\right) dx\, dy\, dz \\ &= \iint_D \left(\frac{\partial F_2}{\partial x}(x, y) - \frac{\partial F_1}{\partial y}(x, y)\right) dx\, dy,\end{aligned}$$

which is the right-hand side of Green's theorem.

Now we compute the left-hand side of (8.1). The boundary, ∂V, of V is the union of

- the flat top, with normal $\mathbf{k}$,
- the flat bottom, with normal $-\mathbf{k}$,
- the side, traced out by the curve C as one slides it from $z = 0$ to $z = 1$.

Since $\mathbf{G} \cdot \mathbf{k} = 0$, we have

$$\iint_{\partial V} \mathbf{G} \cdot d\mathbf{S} = \iint_{\text{top}} \mathbf{G} \cdot \mathbf{k}\, dS + \iint_{\text{bottom}} \mathbf{G} \cdot (-\mathbf{k})\, dS + \iint_{\text{side}} \mathbf{G} \cdot d\mathbf{S} = \iint_{\text{side}} \mathbf{G} \cdot d\mathbf{S}\,.$$

To compute the integral over the side, we parameterize it. Suppose that

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \le t \le b,$$

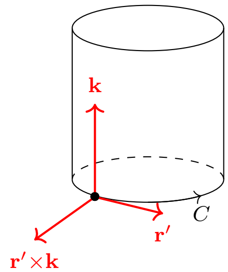
is a parameterization of C. Then

$$\Phi(t, z) = \mathbf{r}(t) + z\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z\mathbf{k}, \quad a \le t \le b,\ 0 \le z \le 1,$$

is a parameterization of the side. We have

$$\Phi_t \times \Phi_z = (x'(t), y'(t), 0) \times (0, 0, 1) = \mathbf{r}'(t) \times \mathbf{k} = y'(t)\mathbf{i} - x'(t)\mathbf{j}.$$

Using the right-hand rule, we see that this gives the outward normal.



Therefore,

$$
\begin{aligned}
\iint\limits_{\partial V} \mathbf{G} \cdot d\mathbf{S} = \iint\limits_{\text{side}} \mathbf{G} \cdot d\mathbf{S} \\
&= \int_a^b \int_0^1 \mathbf{G}(\Phi(t,z)) \cdot (\Phi_t \times \Phi_z)\, dz\, dt \\
&= \int_a^b \int_0^1 \big(F_2(x,y), -F_1(x,y), 0\big) \cdot \big(y'(t), -x'(t), 0\big)\, dz\, dt \\
&= \int_a^b \int_0^1 \big(F_2(x,y)y'(t) + F_1(x,y)x'(t)\big)\, dz\, dt \\
&= \int_a^b \big(F_2(x,y)y'(t) + F_1(x,y)x'(t)\big)\, dt \\
&= \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt \\
&= \oint_C \mathbf{F} \cdot d\mathbf{s}\,.
\end{aligned}
$$

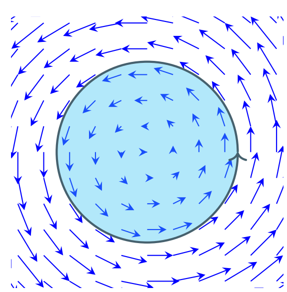
This is precisely the left-hand side of Green's theorem. □

Example 8.3.5. Consider the vector field

$$\mathbf{F}(x,y) = (-y, x)$$

and the unit disk

$$D = \{(x,y) : x^2 + y^2 \leq 1\}.$$



The boundary is the unit circle, which we can parameterize by

$$\mathbf{r}(\theta) = (\cos\theta, \sin\theta), \quad \theta \in [0, 2\pi].$$

Note that this has the correct (that is, counterclockwise) orientation. We have

$$\mathbf{r}'(\theta) = (-\sin\theta, \cos\theta).$$

Thus the left-hand side of Green's theorem is

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta))\cdot\mathbf{r}'(\theta)\,d\theta = \int_0^{2\pi} (-\sin\theta,\cos\theta)\cdot(-\sin\theta,\cos\theta)\,d\theta$$
$$= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

On the other hand, the right-hand side of Green's theorem is

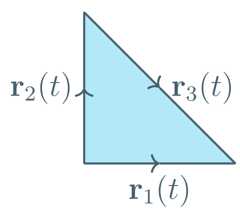
$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_D 2\,dx\,dy = 2\,\mathrm{Area}(D) = 2\pi.$$

So we have verified Green's theorem. △

Example 8.3.6. Let D be the triangle with vertices at $(0,0)$, $(0,1)$, and $(1,0)$. Let's verify Green's theorem on D for

$$\mathbf{F}(x,y) = (x,y).$$

We parameterize the boundary C of D by



$$\begin{aligned} \mathbf{r}_1(t) &= (t,0), & t&\in[0,1], & \mathbf{r}_1'(t) &= (1,0),\\ \mathbf{r}_2(t) &= (0,t), & t&\in[0,1], & \mathbf{r}_2'(t) &= (0,1),\\ \mathbf{r}_3(t) &= (t,1-t), & t&\in[0,1], & \mathbf{r}_3'(t) &= (1,-1). \end{aligned}$$

Note that the pieces $\mathbf{r}_2(t)$ and $\mathbf{r}_3(t)$ are oriented in the wrong direction, so we will need to add a negative sign in front of the corresponding terms. So the left-hand side of Green's theorem is

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{s} &= \int_{\mathbf{r}_1} \mathbf{F}\cdot d\mathbf{s} - \int_{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{s} - \int_{\mathbf{r}_3} \mathbf{F}\cdot d\mathbf{s}\\ &= \int_0^1 (t,0)\cdot(1,0)\,dt - \int_0^1 (0,t)\cdot(0,1)\,dt - \int_0^1 (t,1-t)\cdot(1,-1)\,dt\\ &= \int_0^1 t\,dt - \int_0^1 t\,dt - \int_0^1 (t-1+t)\,dt\\ &= -\left[t^2 - t\right]_{t=0}^1 = 0. \end{aligned}$$

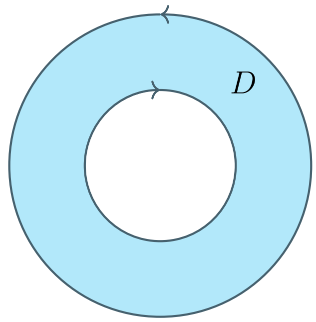
On the other hand, the right-hand side of Green's theorem is

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_D 0\,dx\,dy = 0.$$

So we have verified Green's theorem. △

Example 8.3.7. Consider

$$\mathbf{F}(x,y) = \left(-ye^{x^2+y^2}, xe^{x^2+y^2}\right),$$
$$D = \{(x,y) : x^2+y^2 \in [1,4]\}.$$



The boundary ∂D consists of two circles:

$$\begin{aligned} &\mathbf{r}_1(\theta) = (\cos\theta, \sin\theta), && \theta \in [0, 2\pi], && \text{(wrong orientation)} \\ &\mathbf{r}_2(\theta) = (2\cos\theta, 2\sin\theta), && \theta \in [0, 2\pi]. && \text{(correct orientation)} \end{aligned}$$

We have

$$\mathbf{r}_1'(\theta) = (-\sin\theta, \cos\theta), \qquad \mathbf{r}_2'(\theta) = (-2\sin\theta, 2\cos\theta),$$

and so

$$\begin{aligned} \mathbf{F}(\mathbf{r}_1(\theta)) \cdot \mathbf{r}_1'(\theta) &= (-e\sin\theta, e\cos\theta) \cdot (-\sin\theta, \cos\theta) = e, \\ \mathbf{F}(\mathbf{r}_2(\theta)) \cdot \mathbf{r}_2'(\theta) &= (-2e^4\sin\theta, 2e^4\cos\theta) \cdot (-2\sin\theta, 2\cos\theta) = 4e^4. \end{aligned}$$

Therefore, the left-hand side of Green's theorem is

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{s} = -\int_0^{2\pi} e\, d\theta + \int_0^{2\pi} 4e^4\, d\theta = 8\pi e^4 - 2\pi e.$$

Now let's compute the right-hand side. The scalar curl of $\mathbf{F}$ is

$$\begin{aligned} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial x}\left(xe^{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-ye^{x^2+y^2}\right) \\ &= e^{x^2+y^2} + 2x^2e^{x^2+y^2} + e^{x^2+y^2} + 2y^2e^{x^2+y^2} = 2(1+x^2+y^2)e^{x^2+y^2}. \end{aligned}$$

Using polar coordinates, we have

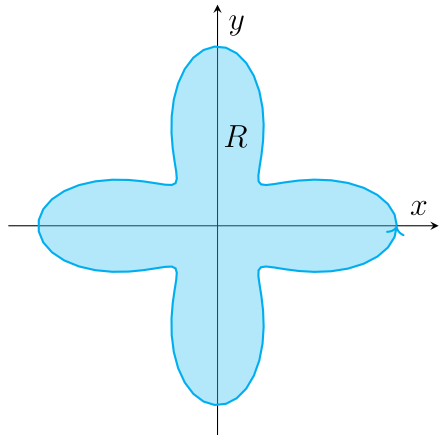
$$\begin{aligned} \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy &= \int_0^{2\pi} \int_1^2 2(1+r^2)e^{r^2} r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_1^4 (1+u)e^u\, du\, d\theta && (u = r^2,\ du = 2r\, dr) \\ &= \int_0^{2\pi} [ue^u]_{u=1}^4\, d\theta \\ &= \int_0^{2\pi} (4e^4 - e)\, d\theta = (4e^4 - e)2\pi = 8\pi e^4 - 2\pi e. \end{aligned}$$

So we have verified Green's theorem. △

Green's theorem can be used to compute the areas of regions bounded by curves, as the next example illustrates.

Example 8.3.8. Let R be the region bounded by the curve

$$\mathbf{r}(t) = \big((2+\cos 4t)\cos t, (2+\cos 4t)\sin t\big), \quad 0 \le t \le 2\pi.$$



If we choose a vector field $\mathbf{F}$ with scalar curl

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1,$$

then, by Green's theorem, we have

$$\text{Area}(R) = \iint\limits_R 1\,dx\,dy = \iint\limits_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s}\,.$$

Let's choose $\mathbf{F}(x,y) = (0,x)$, which does satisfy $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$. So

$$\begin{aligned} \text{Area}(R) &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}) \cdot \mathbf{r}'(t)\,dt \\ &= \int_0^{2\pi} (2+\cos 4t)(\cos t)\big((2-4\sin 4t)\sin t + (2+\cos 4t)\cos t\big)\,dt \\ &= \frac{9\pi}{2}, \end{aligned}$$

where the last integral was computed using techniques from single-variable calculus. (We've omitted the details of this last computation.) △

Warning 8.3.9. In Green's theorem (Theorem 8.3.3), it is crucial that $\mathbf{F}$ is C^2 on *all* of D. For example, with

$$\mathbf{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \quad D = \{(x,y) : x^2+y^2 \le 1\},$$

the equality in Green's theorem does *not* hold. Note that $\mathbf{F}$ is not defined at the origin, which is contained in D. For details, see [FRYe, Examples 4.3.7, 4.3.8].

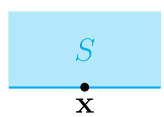
Exercises.

Exercises from [FRYd, §4.3]: Q1–Q25.

8.4 Stokes' theorem

Our final variant of the fundamental theorem of calculus is Stokes' theorem, which is a three-dimensional version of Green's theorem. It relates an integral over a surface in $\mathbb{R}^3$ with an integral over the curve bounding the surface.

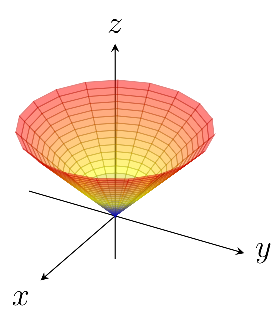
Before stating Stokes', we introduce some terminology and conventions. Suppose S is a piecewise smooth oriented surface in $\mathbb{R}^3$. Remember that this means we have "chosen a side" of S. More precisely, we have chosen a unit normal vector at each point of S in a continuous way. In Definition 1.1.6, we defined the boundary of S to be the set of all points such that any ball around the point contains points of S and points not in S. By this definition, the boundary of most surfaces is the entire surface itself, which is not what we want for Stokes' theorem. So here we say that a point $\mathbf{x} \in S$ is a *boundary point* if, when you zoom in close, it looks as follows:



(This definition can be made more precise with additional work, but boundaries will be clear in all of our examples.) Then the *boundary*, ∂S, of S is the set of all its boundary points. Although we have now used the notation ∂S to denote two different types of boundary, the meaning should be clear from the context. For instance, whenever we speak of the boundary of a surface in $\mathbb{R}^3$, we are using the new definition just introduced here.

Example 8.4.1. For the cone

$$S = \{(x, y, z) : x^2 + y^2 = z^2,\ z \in [0, 1]\}$$

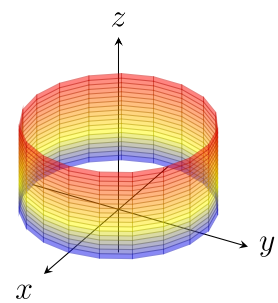


we have

$$\partial S = \{(x, y, 1) : x^2 + y^2 = 1\}. \qquad \triangle$$

Example 8.4.2. For the cylinder

$$S = \{(x, y, z) : x^2 + y^2 = 1,\ z \in [0, 1]\}$$

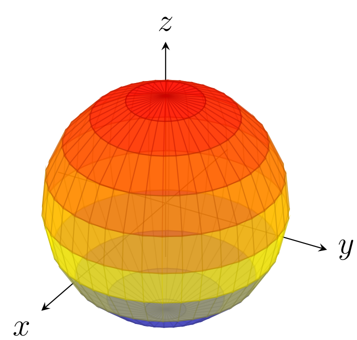


we have

$$\partial S = \{(x, y, z) : x^2 + y^2 = 1,\ z \in \{0, 1\}\}. \qquad \triangle$$

Example 8.4.3. For the sphere

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

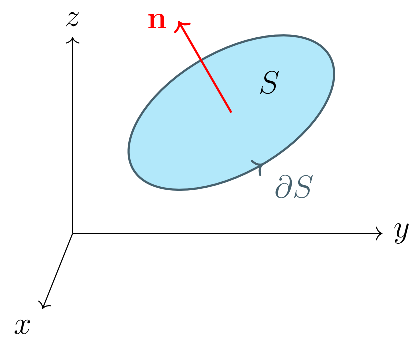


we have $\partial S = \varnothing$. $\triangle$

We will be most interested in surfaces whose boundaries are finite unions of piecewise smooth, simple, closed curves. We want a convention for orienting these boundaries. For Green's theorem (Theorem 8.3.3), we oriented the boundary such that, when you walk along the curve following the orientation, the surface is on your left. We essentially want the same convention for surfaces in $\mathbb{R}^3$, except that now we need to specify where your feet and head are as you walk!

Convention 8.4.4. If S is a piecewise smooth oriented surface in $\mathbb{R}^3$ whose boundary ∂S is a finite union of piecewise smooth, simple, closed curves, then we orient the boundary so that

- if you walk along ∂S in the direction of the orientation of ∂S,
- with the vector from your feet to your head having the same direction as the chosen normal $\mathbf{n}$ for S (giving its orientation),
- then S is on your left-hand side.



Theorem 8.4.5 (Stokes' theorem)**.** *Suppose that S is a bounded piecewise smooth oriented surface whose boundary ∂S is a finite union of piecewise smooth, simple, closed curves, with orientation chosen according to Convention 8.4.4. Furthermore suppose that $\mathbf{F}$ is a C^1 vector field on S. Then*

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint\limits_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}\,.$$

Proof. For a proof of Stokes' theorem, see [FRYe, Th. 4.4.1]. The proof involves reducing to Green's theorem; see Remark 8.4.6. □

Remark 8.4.6. Suppose that $\mathbf{F}$ has zero z-component, so that $\mathbf{F} = (F_1, F_2, 0)$, and that the surface S lies in the xy-plane, so that we have

$$S = \{(x, y, 0) : (x, y) \in D\}$$

for some $D \subseteq \mathbb{R}^2$. Then, using (5.3), we have

$$\iint\limits_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dS = \iint\limits_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy\,.$$

Therefore, in this situation, Stokes' theorem reduces to Green's theorem.

Example 8.4.7. Let's verify Stokes' theorem for

$$\mathbf{F}(x, y, z) = (xy, xy, 0), \qquad S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}.$$

Since $\partial S = \varnothing$, the left-hand side of Stokes' theorem is $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = 0$. So we need to check that the right-hand side is also zero. We parameterize S, which is the unit sphere, using spherical coordinates:

$$\Phi(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi), \quad \theta \in [0, 2\pi],\ \varphi \in [0, \pi].$$

As we computed in Example 7.7.1, we have

$$\Phi_\theta \times \Phi_\varphi = (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi).$$

We have

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z}(xy), \frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x} 0, \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(xy) \right) = (0, 0, y - x),$$

and so

$$\operatorname{curl} \mathbf{F}(\Phi(\theta, \varphi)) = (0, 0, \sin\theta \sin\varphi - \cos\theta \sin\varphi).$$

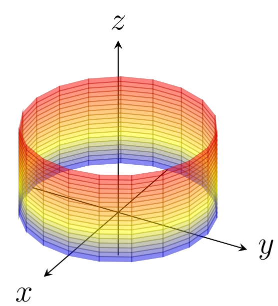
Therefore

$$\begin{aligned}
\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^{\pi} \operatorname{curl} \mathbf{F}(\Phi(\theta, \varphi)) \cdot (\Phi_\theta \times \Phi_\varphi) \, d\varphi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \sin^2\varphi \cos\varphi (\cos\theta - \sin\theta) \, d\varphi \, d\theta \\
&= \int_0^{\pi} \sin^2\varphi \cos\varphi \, [\sin\theta + \cos\theta]_{\theta=0}^{2\pi} \, d\varphi \\
&= 0,
\end{aligned}$$

as expected. △

Example 8.4.8. Let's verify Stokes' theorem for

$$\mathbf{F}(x, y, z) = (-e^z y, e^z x, 0), \qquad S = \{(x, y, z) : x^2 + y^2 = 1, \ z \in [0, 1]\}.$$



We parameterize the cylinder S by

$$\Phi(\theta, z) = (\cos\theta, \sin\theta, z), \quad \theta \in [0, 2\pi], \ z \in [0, 1].$$

We have

$$\begin{aligned}
\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z}(e^z x), \frac{\partial}{\partial z}(-e^z y) - \frac{\partial}{\partial x} 0, \frac{\partial}{\partial x}(e^z x) - \frac{\partial}{\partial y}(-e^z y) \right) \\
&= (-e^z x, -e^z y, 2e^z)
\end{aligned}$$

and

$$\Phi_\theta \times \Phi_z = (-\sin\theta, \cos\theta, 0) \times (0, 0, 1) = (\cos\theta, \sin\theta, 0),$$

which is the outward orientation of the cylinder. Next we compute

$$\begin{aligned}
\nabla \times \mathbf{F}(\Phi(\theta, z)) \cdot (\Phi_\theta \times \Phi_z) &= (-e^z \cos\theta, -e^z \sin\theta, 2e^z) \cdot (\cos\theta, \sin\theta, 0) \\
&= -e^z \cos^2\theta - e^z \sin^2\theta = -e^z.
\end{aligned}$$

Therefore, the right-hand side of Stokes' theorem is

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} (-e^z) \, d\theta \, dz = -2\pi \, [e^z]_{z=0}^1 = 2\pi(1 - e).$$

To compute the left-hand side, we parameterize the two pieces of the boundary of the cylinder:

$$\begin{aligned} \mathbf{r}_1(t) &= (\cos t, \sin t, 1), && t \in [0, 2\pi], && \text{(negative orientation)} \\ \mathbf{r}_2(t) &= (\cos t, \sin t, 0), && t \in [0, 2\pi]. && \text{(positive orientation)} \end{aligned}$$

Then the left-hand side of Stokes' theorem is

$$\begin{aligned} \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= -\int_{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{s} \\ &= -\int_0^{2\pi} \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\,dt + \int_0^{2\pi} \mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t)\,dt \\ &= -\int_0^{2\pi} (-e\sin t, e\cos t, 0) \cdot (-\sin t, \cos t, 0)\,dt \\ &\qquad + \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0)\,dt \\ &= -\int_0^{2\pi} e\,dt + \int_0^{2\pi} dt \\ &= -2\pi e + 2\pi = 2\pi(1-e), \end{aligned}$$

as expected. △

We conclude with some connections between Stokes' theorem and other concepts we have seen in this course. Suppose S_1 and S_2 are two oriented surfaces having the same boundary curve C with orientation chosen according to Convention 8.4.4. (Note that we are assuming that Convention 8.4.4 applied to both S_1 and S_2 yields the same orientation on C.) Then, by Stokes' theorem

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}\,.$$

In other words, the integral of $\operatorname{curl} \mathbf{F}$ is the same over S_1 as is it over S_2.

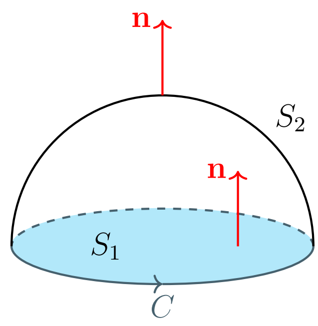
For example, if C is the unit circle

$$C = \{(x, y, 0) : x^2 + y^2 = 1\}$$

oriented counterclockwise when viewed from above, then both

$$\begin{aligned} S_1 &= \{(x, y, z) : x^2 + y^2 \le 1,\ z = 0\} \quad \text{and} \\ S_2 &= \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \ge 0\}, \end{aligned}$$

with upward pointing orientation, have boundary C.



Then, by Stokes' theorem, we have

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}\,. \tag{8.2}$$

In fact, there is another way to see why (8.2) holds. Let

$$V = \{(x, y, z) : x^2 + y^2 + z^2 \le 1,\ z \ge 0\}$$

be the solid between S_1 and S_2. Then

$$\partial V = S_1 \cup S_2.$$

Note that the outward-pointing normal to ∂V is $\mathbf{n}$ on S_2 and $-\mathbf{n}$ on S_1. Thus we have

$$\begin{aligned} \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} - \iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \iint_{\partial V} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\ &= \iiint_V \nabla \cdot (\nabla \times \mathbf{F})\, dV \quad \text{(by the divergence theorem)} \\ &= 0, \end{aligned}$$

where, in the last equality, we used the vector identity $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ from Proposition 8.1.6(a).

Remark 8.4.9. If $\mathbf{F}$ is a C^1 vector field on all of $\mathbb{R}^3$ and satisfies $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere, then Stokes' theorem implies that $\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$ for all closed curves C. Thus, by Theorem 6.5.1, $\mathbf{F}$ is conservative. This gives an alternative proof of Theorem 6.5.2.

Exercises.

Exercises from [FRYd, §4.4]: Q1–Q28.

Bibliography

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