MAT 2122 – Fall 2021

Midterm Exam – Solutions

Professor: Alistair Savage

Your solutions should be submitted through Brightspace in ***.pdf, .jpg, or .png format****. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam has a possible oral component. You may be contacted after the test to arrange a Zoom meeting to explain your solutions. If you are contacted, these explanations are a part of your midterm test, and will be taken into account when determining your grade.

This exam ends at 2:15pm. You may not write anything on your pages after this time. You will then have until 2:25pm to scan and submit your solutions on Brightspace.

QUESTION 1 **(7 pts)**. Consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = e^{y-x}\cos y.$$

(a) Find the degree 1 Taylor polynomial $p_1(h_1,h_2)$ of f at the point $(0,0)$.

Solution: We have

$$f(0,0) = 1, \quad \nabla f(x,y) = \left(-e^{y-x}\cos y, e^{y-x}\cos y - e^{y-x}\sin y\right), \quad \nabla f(0,0) = (-1,1).$$

Thus, the degree 1 Taylor polynomial is

$$p_1(h_1,h_2) = f(0,0) + \nabla f(0,0)\cdot(h_1,h_2) = 1 + (-1,1)\cdot(h_1,h_2) = 1 - h_1 + h_2.$$

(b) Find the degree 2 Taylor polynomial $p_2(h_1,h_2)$ of f at the point $(0,0)$.

Solution: We have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= e^{y-x}\cos y, & \frac{\partial^2 f}{\partial x^2}(0,0) &= 1,\\
\frac{\partial^2 f}{\partial y\,\partial x} &= -e^{y-x}\cos y + e^{y-x}\sin y, & \frac{\partial^2 f}{\partial y\,\partial x}(0,0) &= -1,\\
\frac{\partial^2 f}{\partial y^2} &= e^{y-x}\cos y - e^{y-x}\sin y - e^{y-x}\sin y - e^{y-x}\cos y, & \frac{\partial^2 f}{\partial y^2}(0,0) &= 0,\\
\frac{\partial^2 f}{\partial x\,\partial y} &= -e^{y-x}\cos y + e^{y-x}\sin y, & \frac{\partial^2 f}{\partial x\,\partial y}(0,0) &= -1.
\end{aligned}$$

Thus the Hessian is

$$\mathbf{H}f(0,0) = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}.$$

Therefore, the degree 2 Taylor polynomial is

$$\begin{aligned}
p_2(h_1,h_2) &= p_1(h_1,h_2) + \frac{1}{2}\begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\\
&= 1 - h_1 + h_2 + \frac{1}{2}h_1^2 - h_1h_2.
\end{aligned}$$

QUESTION 2 **(5 pts)**. Is the equation
$$y^2z^3 = e^{xyz}$$
solvable for $z = g(x, y)$ near the point $(x, y, z) = (0, -1, 1)$? If so, compute $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ at $(x, y) = (0, -1)$.

Solution: Let
$$F(x, y, z) = y^2z^3 - e^{xyz}.$$
Then
$$\frac{\partial F}{\partial z} = 3y^2z^2 - xye^{xyz}, \qquad \frac{\partial F}{\partial z}(0, -1, 1) = 3 \neq 0.$$
Thus, by the implicit function theorem, we can solve for $z = g(x, y)$ near $(0, -1, 1)$. We also have
$$\frac{\partial F}{\partial x} = -yze^{xyz}, \qquad \frac{\partial F}{\partial x}(0, -1, 1) = 1,$$
$$\frac{\partial F}{\partial y} = 2yz^3 - xze^{xyz}, \qquad \frac{\partial F}{\partial y}(0, -1, 1) = -2.$$
Hence,
$$\frac{\partial g}{\partial x}(0, -1) = -\frac{\frac{\partial F}{\partial x}(0, -1, 1)}{\frac{\partial F}{\partial z}(0, -1, 1)} = -\frac{1}{3}$$
and
$$\frac{\partial g}{\partial y}(0, -1) = -\frac{\frac{\partial F}{\partial y}(0, -1, 1)}{\frac{\partial F}{\partial z}(0, -1, 1)} = \frac{2}{3}.$$

QUESTION 3 **(5 pts)**. Define $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ by
$$g(x, y, z) = \left(e^{xy}, x^2 + y, x \sin z\right).$$
Suppose also that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function with
$$f(-1, 0, \pi) = -2, \quad f(1, 1, 0) = 5, \quad \nabla f(-1, 0, \pi) = (2, 1, 0), \quad \text{and} \quad \nabla f(1, 1, 0) = (5, 1, 3).$$
Compute $\nabla(f \circ g)(-1, 0, \pi)$.

Solution: We first compute
$$Dg(x, y, z) = \begin{bmatrix} ye^{xy} & xe^{xy} & 0 \\ 2x & 1 & 0 \\ \sin z & 0 & x \cos z \end{bmatrix}, \qquad Dg(-1, 0, \pi) = \begin{bmatrix} 0 & -1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
Note also that $g(-1, 0, \pi) = (1, 1, 0)$. Since f and g are both differentiable functions, we have, by the chain rule,
$$\begin{aligned} D(f \circ g)(-1, 0, \pi) &= Df(g(-1, 0, \pi))Dg(-1, 0, \pi) \\ &= Df(1, 1, 0)Dg(-1, 0, \pi) \\ &= \begin{bmatrix} 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -4 & 3 \end{bmatrix}. \end{aligned}$$
Thus,
$$\nabla(f \circ g)(-1, 0, \pi) = (-2, -4, 3).$$

QUESTION 4 **(8 pts)**. Determine whether the function

$$f(x,y) = x^2 + y^2 - 2x - 4y$$

has a minimum and maximum on the set

$$S = \{(x,y) : x^2 + y^2 \leq 20\}.$$

If so, determine the minimum and maximum values and the points at which they occur.

Solution: The function f is continuous, and the set S is a disk, hence it is bounded and contains its boundary. Therefore, by the extreme value theorem, f attains a global maximum and global minimum.

On the interior, $S \setminus \partial S$, we look for critical points using the first derivative test. We have $\nabla f = (2x-2, 2y-4)$. Hence, $\nabla f = (0,0)$ implies $(x,y) = (1,2)$. This point is contained in the set S, since

$$1^2 + 2^2 = 5 \leq 20.$$

On the boundary, we use the Lagrange multiplier method. Set $g(x,y) := x^2 + y^2$, so that

$$\partial S = \{(x,y) : g(x,y) = 20\}.$$

We have $\nabla g = (2x, 2y)$.

The critical points satisfy either $\nabla g = \mathbf{0}$ or $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$. In the first case, we have $x = y = 0$; however, since $0^2 + 0^2 = 0 \neq 20$, this point is not on the boundary.

In the second case, we obtain the equations

$$\begin{aligned} 2x - 2 &= 2\lambda x, \\ 2y - 4 &= 2\lambda y, \\ x^2 + y^2 &= 20. \end{aligned}$$

Since $\lambda = 1$ would give $-2 = 0$ in the first equation, we may assume $\lambda \neq 1$. Thus, solving for x and y in the first two equations gives

$$y = \frac{2}{1-\lambda} = 2x.$$

Substituting this into the third equation yields

$$\begin{aligned} x^2 + (2x)^2 &= 20 \\ \implies x^2 &= 20/5 = 4 \\ \implies x &= \pm 2 \end{aligned}$$

Using $y = 2x$, we see that this gives two critical points: $\pm(2,4)$.

We compute

$$f(1,2) = -5, \quad f(2,4) = 0, \quad f(-2,-4) = 40.$$

Thus the maximum is 40, obtained at $(-2,-4)$, and the minimum is -5, obtained at $(1,2)$.

QUESTION 5 **(5 pts)**. Let

$$W = \{(x,y,z) : x^2 + y^2 + z^2 \leq 4,\ z \geq 0\} \subseteq \mathbb{R}^3.$$

Compute

$$\iiint\limits_W \cos\left(\left(\sqrt{x^2+y^2+z^2}\right)^3\right) dx\, dy\, dz\,.$$

Solution: The region W is the half of the ball centred at the origin of radius 2 lying above the xy-plane. In spherical coordinates, this region corresponds to

$$W^* = \left\{(\rho, \theta, \varphi) : 0 \leq \rho \leq 2,\ 0 \leq \varphi \leq \tfrac{\pi}{2},\ 0 \leq \theta \leq 2\pi\right\}.$$

Thus,

$$\begin{aligned}
\iiint\limits_W \cos\left(\left(\sqrt{x^2+y^2+z^2}\right)^3\right) dV &= \iiint\limits_W \cos(\rho^3)\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi \\
&= \int_0^{\pi/2} \int_0^{2\pi} \int_0^2 \cos(\rho^3)\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi \\
&= \int_0^{\pi/2} \int_0^{2\pi} (\sin\varphi) \left[\frac{1}{3}\sin\left(\rho^3\right)\right]_{\rho=0}^2 d\theta \, d\varphi \\
&= \frac{\sin(8)}{3} \int_0^{\pi/2} \int_0^{2\pi} \sin\varphi \, d\theta \, d\varphi \\
&= \frac{2\pi\sin(8)}{3} \int_0^{\pi/2} \sin\varphi \, d\varphi \\
&= \frac{2\pi\sin(8)}{3} \left[-\cos\varphi\right]_{\varphi=0}^{\pi/2} \\
&= \frac{2\pi\sin(8)}{3}.
\end{aligned}$$