MAT 2122 – Fall 2021

Final Exam

Professor: Alistair Savage

*Your solutions should be submitted through Brightspace in **.pdf, .jpg, or .png format**. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam has a possible oral component. You may be contacted after the test to arrange a Zoom meeting to explain your solutions. If you are contacted, these explanations are a part of your midterm test, and will be taken into account when determining your grade.

This exam ends at 12:30pm. You may not write anything on your pages after this time. You will then have until 12:40pm to scan and submit your solutions on Brightspace. You must remain on camera until your work is submitted.

If you wish to leave the exam early, you must request permission by sending a private message to the instructor in the Zoom chat. Once permission is granted, you may not write anything further on your pages, and you have 10 minutes to scan and submit your solutions.

QUESTION 1 **(3 pts)**. Consider the function $f(x, y) = e^x \sin(y) + e^y \sin(x)$.

(a) At the point $(0, \pi)$, in which direction is f increasing the fastest? Give your answer as a unit vector.
(b) Compute the directional derivative of $f(x, y)$ at the point $(0, \pi)$ in the direction $\mathbf{v} = \left(\frac{3}{5}, -\frac{4}{5}\right)$.

QUESTION 2 **(5 pts)**. Consider the function

$$f(x, y) = xe^y + x^3 - y^2 - 4x - xy.$$

(a) Find all critical points of the form $(x, 0)$ for this function.
(b) For each critical point found in (a), determine whether it is a local minimum, local maximum, or saddle point.

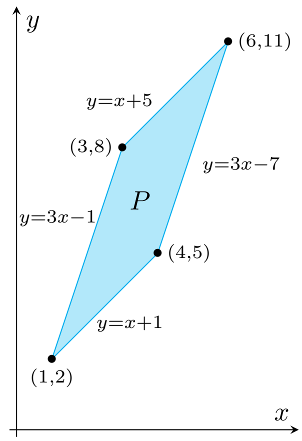
QUESTION 3 **(7 pts)**. Consider the set

$$S = \{(x, y) : x^4 + y^2 = 2\}.$$

Which point(s) on S are closest to the origin? Which point(s) are the furthest away from the origin? In other words, find the points where $f(x, y) = x^2 + y^2$, restricted to S, attains its global extrema.

QUESTION 4 **(6 pts)**. Let P be the parallelogram bounded by

$$y = 3x - 1, \quad y = 3x - 7, \quad y = x + 1, \quad y = x + 5.$$



Introduce the variables

$$u = y - 3x \quad \text{and} \quad v = y - x.$$

(a) Solve for x and y in terms of u and v. More precisely, define a function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $(x, y) = T(u, v)$.
(b) Compute the Jacobian determinant $\frac{\partial(x,y)}{\partial(u,v)}$.
(c) Find the subset $P^* \subseteq \mathbb{R}^2$ such that $T(P^*) = P$.
(d) Compute $\iint\limits_P (y - x)\,dx\,dy$.

Question 5 **(5 pts)**. Fix $k \in \mathbb{R}$, and consider the vector field

$$\mathbf{F}(x, y) = (\cos(x) + \cos(x)\cos(y), k\sin(x)\sin(y) + k\sin(y)).$$

(a) For which value(s) of k is this vector field conservative?
(b) For the value(s) of k found in part (a), find a potential for F.

Question 6 **(4 pts)**. Consider a wire parameterized by

$$\mathbf{r}(t) = (\sin(t), 2t, \cos(t)), \quad 0 \le t \le \pi.$$

Suppose the density of the wire at the point (x, y, z) is $x + y + z$. Compute the mass of the wire.

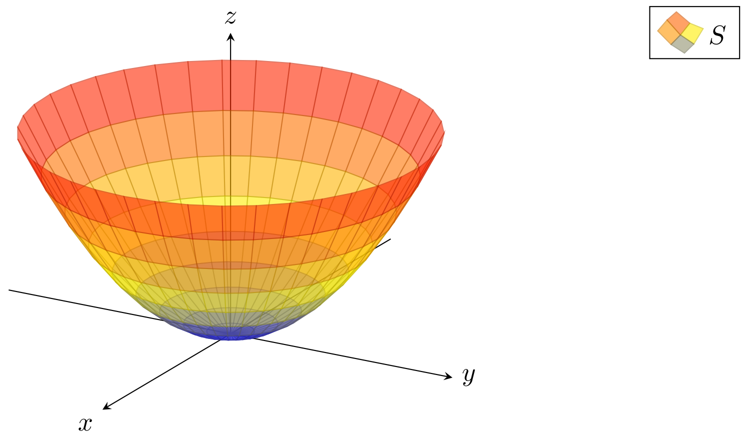
Question 7 **(5 pts)**. The flow of a fluid has velocity described by the vector field

$$\mathbf{F}(x, y, z) = (y, -x, 1 - z).$$

Determine the net flow rate upward through the surface

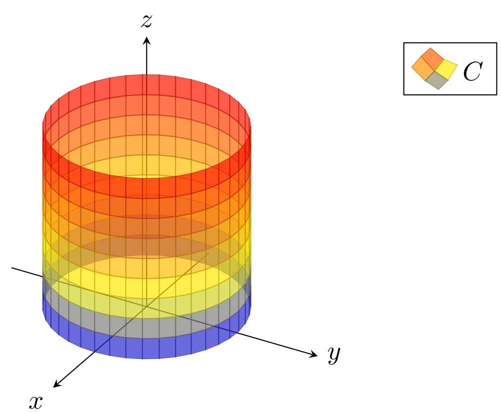
$$S = \{(x, y, z) : z = x^2 + y^2 \le 1\}.$$

(If the net flow is downward, then your answer should be negative.)



Question 8 **(6 pts)**. Consider the tube

$$C = \{(x, y, z) : x^2 + y^2 = 1,\ 0 \le z \le 1\},$$



oriented with normal pointing inwards (i.e. towards z-axis), and define the vector field

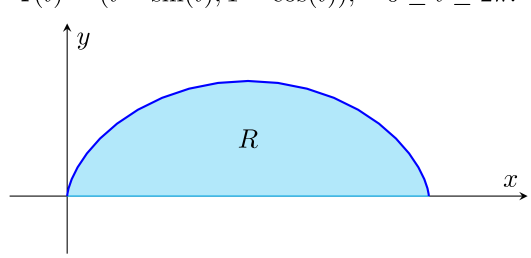
$$\mathbf{F}(x, y, z) = \left(x^3 e^z + 2xy + z\cos(y^2), y^3 e^z + \sin(xe^z) - y^2, 0\right).$$

Use Gauss' divergence theorem to compute

$$\iint_C \mathbf{F} \cdot d\mathbf{S}.$$

Question 9 **(5 pts)**. Let R be the region bounded by the x-axis and the curve

$$\mathbf{r}(t) = (t - \sin(t), 1 - \cos(t)), \quad 0 \le t \le 2\pi.$$



(a) Compute the scalar curl of the vector field $\mathbf{F}(x, y) = (y, 0)$.
(b) Use Green's theorem to compute the area of R. *Hint*: Use the vector field $\mathbf{F}$. It may also be helpful to remember the identity $\cos^2(t) = (1 + \cos(2t))/2$.