MAT 2122 – Fall 2021

Final Exam – Solutions

Professor: Alistair Savage

*Your solutions should be submitted through Brightspace in **.pdf, .jpg, or .png format**. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam has a possible oral component. You may be contacted after the test to arrange a Zoom meeting to explain your solutions. If you are contacted, these explanations are a part of your midterm test, and will be taken into account when determining your grade.

This exam ends at 12:30pm. You may not write anything on your pages after this time. You will then have until 12:40pm to scan and submit your solutions on Brightspace. You must remain on camera until your work is submitted.

If you wish to leave the exam early, you must request permission by sending a private message to the instructor in the Zoom chat. Once permission is granted, you may not write anything further on your pages, and you have 10 minutes to scan and submit your solutions.

QUESTION 1 **(3 pts)**. Consider the function $f(x, y) = e^x \sin(y) + e^y \sin(x)$.

(a) At the point $(0, \pi)$, in which direction is f increasing the fastest? Give your answer as a unit vector.
(b) Compute the directional derivative of $f(x, y)$ at the point $(0, \pi)$ in the direction $\mathbf{v} = \left(\frac{3}{5}, -\frac{4}{5}\right)$.

Solution:

(a) We have
$$\nabla f = (e^x \sin(y) + e^y \cos(x), e^x \cos(y) + e^y \sin(x)), \qquad \nabla f(0, \pi) = (e^\pi, -1).$$
Thus, at the point $(0, \pi)$, f is increasing fastest in the direction
$$\frac{1}{\sqrt{e^{2\pi} + 1}}(e^\pi, -1).$$

(b) We compute
$$D_{\mathbf{v}} f(0, \pi) = \nabla f(0, \pi) \cdot \mathbf{v} = (e^\pi, -1) \cdot \left(\frac{3}{5}, -\frac{4}{5}\right) = \frac{3e^\pi + 4}{5}.$$

QUESTION 2 **(5 pts)**. Consider the function
$$f(x, y) = xe^y + x^3 - y^2 - 4x - xy.$$

(a) Find all critical points of the form $(x, 0)$ for this function.

Solution: Compute
$$\frac{\partial f}{\partial x} = e^y + 3x^2 - 4 - y, \qquad \frac{\partial f}{\partial y} = xe^y - 2y - x.$$
We solve for when these are 0, subject to the constraint $y = 0$:
$$0 = \frac{\partial f}{\partial x}(x, 0) = 1 + 3x^2 - 4 = 3(x^2 - 1),$$
$$0 = \frac{\partial f}{\partial y}(x, 0) = x - x = 0.$$
From the first equation, we see that there are exactly two solutions: $(\pm 1, 0)$.

(b) For each critical point found in (a), determine whether it is a local minimum, local maximum, or saddle point.

Solution: We compute

$$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial x \partial y} = e^y - 1, \qquad \frac{\partial^2 f}{\partial y^2} = xe^y - 2.$$

At $(1, 0)$, these are $6, 0, -1$, so the Hessian matrix is

$$H(1,0) = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since the determinant is $-6 < 0$ (or alternatively, by looking at the eigenvalues), we see that this is a saddle point by the second derivative test.

At $(-1, 0)$, the second partial derivatives are $-6, 0, -3$, so the Hessian matrix is

$$H(-1,0) = \begin{bmatrix} -6 & 0 \\ 0 & -3 \end{bmatrix}.$$

Since the determinant is $18 > 0$ and the $(1,1)$-entry is $-6 < 0$ (or again, alternatively, by looking at the eigenvalues), this is a local maximum by the second derivative test

Question 3 **(7 pts)**. Consider the set

$$S = \{(x, y) : x^4 + y^2 = 2\}.$$

Which point(s) on S are closest to the origin? Which point(s) are the furthest away from the origin? In other words, find the points where $f(x, y) = x^2 + y^2$, restricted to S, attains its global extrema.

Solution: Let $g(x, y) = x^4 + y^2$. Since S is compact (closed and bounded) and f is continuous, the function f attains both a maximum and a minimum on S. By the Lagrange multiplier theorem, the extrema of f on S occur at points where ∇f and ∇g are colinear (linearly dependent). We compute

$$\nabla f = (2x, 2y), \qquad \nabla g = (4x^3, 2y).$$

We have $\nabla g = \mathbf{0}$ only when $(x, y) = (0, 0)$, which is not a point of S. Thus, it is enough to solve the equation $\nabla f = \lambda \nabla g$ on S. This means solving the system of equations:

$$2x = 4\lambda x^3 \tag{1}$$
$$2y = 2\lambda y \tag{2}$$
$$2 = x^4 + y^2 \tag{3}$$

Equation (2) implies that either $y = 0$ or $\lambda = 1$. If $y = 0$, then (3) gives $x^4 = 2$ hence $x = \pm\sqrt[4]{2}$. This yields the two critical points $(\pm\sqrt[4]{2}, 0)$.

On the other hand, if $\lambda = 1$, then (1) gives

$$x = 2x^3 \implies 0 = 2x^3 - x = x(2x^2 - 1) \implies x = 0 \text{ or } 2x^2 = 1.$$

In the first case, (3) then implies $y = \pm\sqrt{2}$ and we find the two critical points $(0, \pm\sqrt{2})$.

In the final case, where $2x^2 = 1$, we have $x = \frac{\pm 1}{\sqrt{2}}$, hence $x^4 = \frac{1}{4}$. Then, by (3), $y^2 = 2 - \frac{1}{4} = \frac{7}{4}$, which gives $y = \pm\frac{\sqrt{7}}{2}$. So we get four solutions $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{7}}{2}\right)$.

At each critical point, we compute the value of $f(x, y) = x^2 + y^2$ and we find:

$$f(\pm\sqrt[4]{2}, 0) = \sqrt{2}, \qquad f(0, \pm\sqrt{2}) = 2, \qquad f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{7}}{2}\right) = \frac{1}{2} + \frac{7}{4} = \frac{9}{4}.$$

Hence the maximum value of $f(x, y)$ is $\frac{9}{4}$ and the points furthest away from the origin are

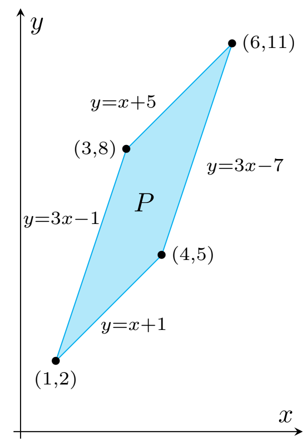
$$\left(\frac{1}{\sqrt{2}}, \frac{\sqrt{7}}{2}\right), \quad \left(\frac{1}{\sqrt{2}}, -\frac{\sqrt{7}}{2}\right), \quad \left(-\frac{1}{\sqrt{2}}, \frac{\sqrt{7}}{2}\right), \quad \left(-\frac{1}{\sqrt{2}}, -\frac{\sqrt{7}}{2}\right).$$

The minimum value of f on S is $\sqrt{2}$ and the closest points to the origin are

$$(\sqrt[4]{2}, 0), \qquad (-\sqrt[4]{2}, 0).$$

QUESTION 4 **(6 pts)**. Let P be the parallelogram bounded by

$$y = 3x - 1, \quad y = 3x - 7, \quad y = x + 1, \quad y = x + 5.$$



Introduce the variables

$$u = y - 3x \quad \text{and} \quad v = y - x.$$

(a) Solve for x and y in terms of u and v. More precisely, define a function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $(x, y) = T(u, v)$.

Solution: We have

$$(x, y) = T(u, v) = T\left(\frac{v-u}{2}, \frac{3v-u}{2}\right).$$

(b) Compute the Jacobian determinant $\frac{\partial(x,y)}{\partial(u,v)}$.

Solution: We have

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}.$$

(c) Find the subset $P^* \subseteq \mathbb{R}^2$ such that $T(P^*) = P$.

Solution: Since the region P is given by

$$-7 \leq y - 3x \leq -1, \qquad 1 \leq y - x \leq 5,$$

we have

$$P^* = [-7, -1] \times [1, 5].$$

(d) Compute $\iint\limits_P (y-x)\,dx\,dy$.

Solution: Using the given change of variables, we compute

$$\iint\limits_P (y-x)\,dx\,dy = \iint\limits_{P^*} v\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_1^5\int_{-7}^{-1}\frac{v}{2}\,du\,dv = \int_1^5 3v\,dv = \left[\frac{3v^2}{2}\right]_{v=1}^5 = \frac{3}{2}(25-1) = 36.$$

QUESTION 5 **(5 pts)**. Fix $k \in \mathbb{R}$, and consider the vector field

$$\mathbf{F}(x,y) = (\cos(x)+\cos(x)\cos(y), k\sin(x)\sin(y)+k\sin(y)).$$

(a) For which value(s) of k is this vector field conservative?
(b) For the value(s) of k found in part (a), find a potential for F.

Solution:

(a) We first compute the scalar curl of $\mathbf{F}$:
$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = k\cos(x)\sin(y) + \cos(x)\sin(y).$$
This is 0 for all x and y if and only if $k=-1$. Since $\mathbf{F}$ is defined on all of $\mathbb{R}^2$, this means that $\mathbf{F}$ is conservative if and only if $k=-1$.

(b) Let $k=-1$. We want to find a function ϕ such that $\nabla\phi = \mathbf{F}$. Thus, we want
$$\frac{\partial\phi}{\partial x} = \cos(x)+\cos(x)\cos(y), \qquad \frac{\partial\phi}{\partial y} = -\sin(x)\sin(y) - \sin(y).$$
Integrating the first equation with respect to x, we have
$$\phi(x,y) = \sin(x)+\sin(x)\cos(y)+\psi(y),$$
where $\psi(y)$ is a function of y only (so that $\frac{\partial\psi}{\partial x}=0$). Differentiating with respect to y now gives
$$\begin{aligned} -\sin(x)\sin(y)+\psi'(y) &= -\sin(x)\sin(y)-\sin(y), \\ \implies \psi'(y) &= -\sin(y), \\ \implies \psi(y) &= \cos(y)+C, \end{aligned}$$
where $C\in\mathbb{R}$. Therefore, we have
$$\phi = \sin(x)+\sin(x)\cos(y)+\cos(y)+C.$$

Alternate Solution: Instead of using the curl test, one can directly start by looking for a potential:
$$\frac{\partial\phi}{\partial x} = \cos(x)+\cos(x)\cos(y), \qquad \frac{\partial\phi}{\partial y} = k\sin(x)\sin(y)+k\sin(y).$$
As above, we integrate the first equation with respect to x to get
$$\phi(x,y) = \sin(x)+\sin(x)\cos(y)+\psi(y).$$
Then, differentiating with respect to y gives
$$-\sin(x)\sin(y)+\psi'(y) = k\sin(x)\sin(y)+k\sin(y),$$
and so
$$\psi'(y) - k\sin(y) = (k+1)\sin(x)\sin(y).$$
Since the left-hand side of the above equation does not depend on x, the right hand side cannot depends on x either. This forces $k+1=0$, hence we find that $k=-1$. Then
$$\psi'(y)+\sin(y) = 0 \implies \psi(y) = \cos(y),$$

recovering the same solution as above.

QUESTION 6 **(4 pts)**. Consider a wire parameterized by

$$\mathbf{r}(t) = (\sin(t), 2t, \cos(t)), \quad 0 \le t \le \pi.$$

Suppose the density of the wire at the point (x, y, z) is $x + y + z$. Compute the mass of the wire.

Solution: We have

$$\mathbf{r}'(t) = (\cos(t), 2, -\sin(t)),$$

and so

$$\|\mathbf{r}'(t)\| = \sqrt{\cos^2(t) + 4 + \sin^2(t)} = \sqrt{5}.$$

Thus the mass is

$$\int_0^\pi (\sin(t) + 2t + \cos(t))\sqrt{5}\, dt = \sqrt{5}\left[-\cos(t) + t^2 + \sin(t)\right]_0^\pi = \sqrt{5}(\pi^2 + 2).$$

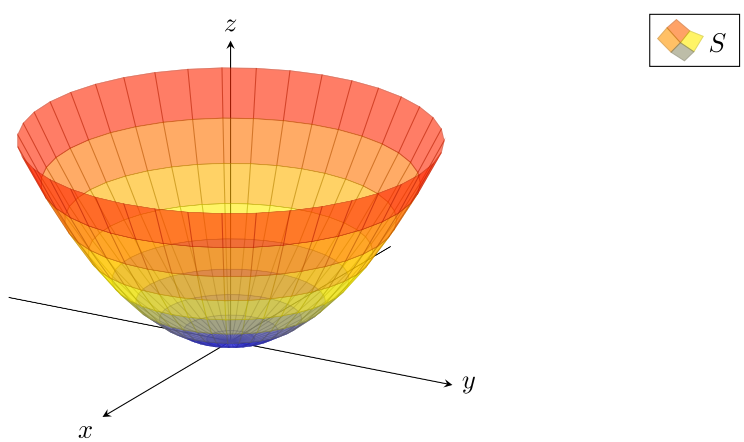
QUESTION 7 **(5 pts)**. The flow of a fluid has velocity described by the vector field

$$\mathbf{F}(x, y, z) = (y, -x, 1 - z).$$

Determine the net flow rate upward through the surface

$$S = \{(x, y, z) : z = x^2 + y^2 \le 1\}.$$

(If the net flow is downward, then your answer should be negative.)



Solution: Parametrize S by

$$T(r, \theta) = (r\cos(\theta), r\sin(\theta), r^2), \qquad r \in [0, 1], \quad \theta \in [0, 2\pi].$$

Then we have

$$\begin{aligned} T_r \times T_\theta &= (\cos(\theta), \sin(\theta), 2r) \times (-r\sin(\theta), r\cos(\theta), 0) \\ &= (-2r^2\cos(\theta), 2r^2\sin(\theta), r\sin^2(\theta)r\cos^2(\theta)) \\ &= (-2r^2\cos(\theta), -2r^2\sin(\theta), r). \end{aligned}$$

Since the z-coordinate is positive, T is oriented upwards, and so the net flow in the upward direction is

$$\iint_T \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 \mathbf{F}(T(\theta, r)) \cdot (T_r \times T_\theta)\, dr\, d\theta$$

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^1 (r\sin(\theta), -r\cos(\theta), 1-r^2)\cdot(-2r^2\cos(\theta), -2r^2\sin(\theta), r)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \left(-2r^3\sin(\theta)\cos(\theta) + 2r^3\cos(\theta)\sin(\theta) + r(1-r^2)\right)dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta \\
&= 2\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_{r=0}^1 \\
&= \frac{\pi}{2}.
\end{aligned}
$$

Alternate Solution: Alternatively, we can parameterize S as the graph of a function, that is as

$$T(x,y) = (x, y, x^2+y^2), \qquad (x,y)\in D = \{(x,y) : x^2+y^2\le 1\}.$$

Then we have

$$T_x \times T_y = (-2x, -2y, 1).$$

Since the z-coordinate is positive, T is already oriented upward, and so the net flow in the upward direction is

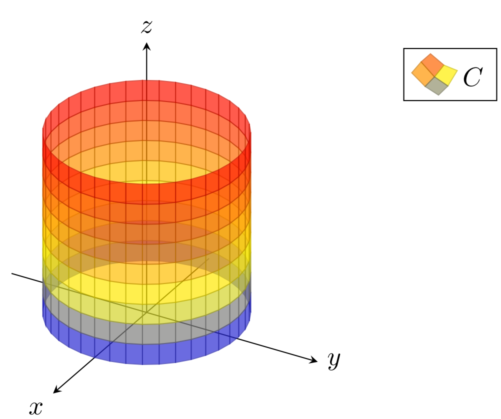
$$
\begin{aligned}
\iint_T \mathbf{F}\cdot d\mathbf{S} &= \iint_D (y, -x, 1-x^2-y^2)\cdot(-2x,-2y,1)\,dx\,dy \\
&= \iint_D \left(-2xy + 2yx + 1 - x^2 - y^2\right)dx\,dy \\
&= \iint_D \left(1-x^2-y^2\right)dx\,dy\,.
\end{aligned}
$$

We now change to polar coordinates, so that $x^2+y^2=r^2$ and $dx\,dy = r\,dr\,d\theta$, to get

$$\iint_D \left(1-x^2-y^2\right)dx\,dy = \int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta \; = 2\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_{r=0}^1 = \frac{\pi}{2}.$$

Question 8 **(6 pts)**. Consider the tube

$$C = \{(x,y,z) : x^2+y^2 = 1,\ 0\le z\le 1\},$$



oriented with normal pointing inwards (i.e. towards z-axis), and define the vector field

$$\mathbf{F}(x,y,z) = \left(x^3e^z + 2xy + z\cos(y^2), y^3e^z + \sin(xe^z) - y^2, 0\right).$$

Use Gauss' divergence theorem to compute

$$\iint_C \mathbf{F}\cdot d\mathbf{S}.$$

Solution: Let A be the solid cylinder,

$$A = \{(x,y,z) : x^2+y^2 \le 1,\ 0 \le z \le 1\},$$

and let S_0, S_1 be the bottom and top of A, so that

$$S_i = \{(x,y,i) : x^2+y^2 \le 1\}.$$

Orient S_0 with normal pointing downward and S_1 with normal pointing upwards. Then $\partial A = S_0 \cup S_1 \cup C$. The positive orientation on ∂A agrees with the prescribed orientations of S_0 and S_1, but not C. Therefore, by the divergence theorem,

$$\iiint_A \nabla\cdot\mathbf{F}\,dV = \iint_{\partial A}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_0}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} - \iint_C \mathbf{F}\cdot d\mathbf{S}.$$

We compute

$$\nabla\cdot\mathbf{F} = 3x^2e^z + 2y + 3y^2e^z - 2y = 3(x^2+y^2)e^z.$$

We note that the normals to S_0 and S_1 are orthogonal to $\mathbf{F}$, and therefore $\iiint_{S_i}\mathbf{F}\cdot d\mathbf{S} = 0$ for $i = 0,1$. So

$$\iint_C \mathbf{F}\cdot d\mathbf{S} = -\iiint_A \nabla\cdot\mathbf{F}\,dV = -\iiint_A 3(x^2+y^2)e^z\ dx\,dy\,dz.$$

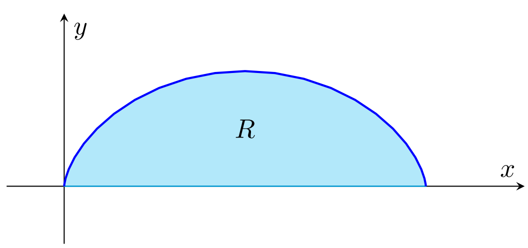
To calculate the right-hand side, we use cylindrical coordinates, with $\theta\in[0,2\pi]$, $z\in[0,1]$, and $r\in[0,1]$. The integrand becomes $3r^2e^z$, and so

$$\iint_C \mathbf{F}\cdot d\mathbf{S} = -\int_0^{2\pi}\int_0^1\int_0^1 3r^3e^z\,dz\,dr\,d\theta = -2\pi\int_0^1 \left[3r^3e^z\right]_{z=0}^1 dr$$

$$= -2\pi\int_0^1 3r^3(e-1)\,dr = -2\pi(e-1)\left[\frac{3r^4}{4}\right]_{r=0}^1 = \frac{3\pi(1-e)}{2}.$$

QUESTION 9 **(5 pts)**. Let R be the region bounded by the x-axis and the curve

$$\mathbf{r}(t) = (t - \sin(t), 1-\cos(t)), \quad 0 \le t \le 2\pi.$$



(a) Compute the scalar curl of the vector field $\mathbf{F}(x,y) = (y,0)$.

Solution: The scalar curl of $\mathbf{F}$ is

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -1.$$

(b) Use Green's theorem to compute the area of R. *Hint*: Use the vector field $\mathbf{F}$. It may also be helpful to remember the identity $\cos^2(t) = (1 + \cos(2t))/2$.

Solution: Let C be the portion of the x-axis from $x = 0$ to $x = 2\pi$, oriented to the right. Note that the curve $\mathbf{r}(t)$ is oriented to the right, which is opposite the orientation that appears in the statement of Green's theorem. Also note that $\mathbf{F}(x, 0) = \mathbf{0}$, so that $\mathbf{F}$ is zero on C. Thus, using Green's theorem, we have

$$\text{Area}(R) = \iint\limits_R dA = -\iint\limits_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = -\int_C \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s}$$

$$= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} (1 - \cos(t), 0) \cdot (1 - \cos(t), \sin(t))\, dt = \int_0^{2\pi} (1 - 2\cos(t) + \cos^2(t))\, dt$$

$$= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos(t) + \frac{1}{2}\cos(2t)\right) dt = \left[\frac{3}{2}t - 2\sin(t) + \frac{1}{4}\sin(2t)\right]_0^{2\pi} = 3\pi.$$