Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2122 – Final exam **Instructors: T. Schmah and A. Tikuisis**

Final Exam (Written part). Due Sunday December 20^{th} 5pm (EST).

Please upload your exam solutions as a single pdf file including your signed statement. Give yourself time to upload the exam. Exams submitted late will be penalized. You may submit multiple times – and this is recommended to avoid missing the deadline. Only the latest submission will be marked.

Examination rules: You are **not** permitted to discuss the exam with any other person during the exam, except that you may email your instructor in case of a problem. You *are* permitted to consult textbooks and other resources, and calculators, including online calculators, except that you are **not** permitted to use any software to provide step-by-step solutions. You are **not** permitted to show an exam question to another person, including posting it on social networks or "study" sites, or to read another person's solution to an exam question.

Failure to follow these rules constitutes academic fraud. Note that if one student copies from the other, both students have committed academic fraud.

Write out the following statement, sign it (below the statement), and include it as part of your exam submission: "I have read the examination rules. My solutions are entirely my own work." Only submissions containing this signed statement will be graded.

[15pts total] **1.** Let $f(x,y,z) := xy^2z^3$, and let $g : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 function such that:

$$g(3,2,-1) = 4,\ \nabla g(3,2,-1) = (3,4,5),$$
$$Hg(3,2,-1) = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 0 & 7 \\ -1 & 7 & 4 \end{bmatrix},$$
$$g(5,5,5) = 4,\ \nabla g(5,5,5) = (0,0,0),$$
$$Hg(5,5,5) = \begin{bmatrix} -5 & 2 & -1 \\ 2 & -1 & 0 \\ -1 & 0 & -4 \end{bmatrix}.$$

[2pts] (a) Compute $DF(3,2,-1)$, where $F : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $F := (f,g)$.

[2pts] (b) Is F in part (a) differentiable at $(3,2,-1)$? *Justify your answer.*

[3pts] (c) Compute $(g \circ c)'(1)$, where $c(t) := (3t, 1+t, -t^2)$.

[3pts] (d) Compute the second order Taylor polynomial of g at $(3,2,-1)$. Write your answer as a polynomial, without using matrices.

[1pt] (e) Calculate the determinant of $Hg(5,5,5)$.

[2pts] (f) Classify the critical point $(5,5,5)$ of g as a local minimum, local maximum, or saddle point. *Justify your answer.*

[2pts] (g) Consider the level set $g(x,y,z) = 4$. Is it possible, near $(3,2,-1)$, to describe this set as the graph of a C^1 function $y = k(x,z)$? *Justify your answer.*

[11pts total] **2.** Let $h(x,y) := x^2 + xy + y^2$ and $D := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

[1pt] (a) What is the boundary of D? *No justification required.*

[2pts] (b) Explain, by quoting a theorem and checking that it applies here, why we know that h attains global maximum and minimum values on the set D.

[8pts] (c) Find the maximum and minimum values of h subject to the constraint $x^2 + y^2 \leq 1$.

[5pts] **3.** Evaluate the integral $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{1+x^3} \, dx\, dy$.
[*Hint. Change the order of integration.*]

[8pts total] **4.** Let $S := \{(x,y) \in \mathbb{R}^2 : 1 \leq xy \leq 2 \text{ and } 2x^2 \leq y \leq 3x^2\}$.

[3pts] (a) Show that the map $\Phi : S \to \mathbb{R}^2$ given by $(u,v) = \Phi(x,y) := (xy, y/x^2)$ is injective and find its image $S^* = \Phi(S)$.

[5pts] (b) Show that the area of S is $\frac{1}{3}(\ln 3 - \ln 2)$.

[7pts total] **5.** Define a path $p : [-2,2] \to \mathbb{R}^2$ by

$$p(t) := (t^3 - 4t, t^2 + t^4).$$

[3pts] (a) Show that p defines a simple closed curve.

[1pt] (b) Find a vector field $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\operatorname{scurl}(\vec{F}) = 1$.

[3pts] (c) Let A denote the region enclosed by the simple closed curve given by p. Use Green's Theorem to show that the area of A is $\frac{7936}{105}$.

[11pts total] **6.** Define $A := \{(x,y,z) : x + x^2 + y^2 - 1 \leq z \leq x\}$ and define $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\vec{F}(x,y,z) := (0, y, z - x).$$

[3pts] (a) Let ∂A^+ denote the boundary of A with the positive orientation (i.e., outward-pointing normals). Determine ∂A and find two parametrized surfaces whose union is ∂A. Determine whether your parametrizations have positive or negative orientations.
[*Hint. Note that in A we have $x + x^2 + y^2 - 1 \leq x$.*]

[4pts] (b) Using the parametrizations from part (a), show directly that

$$\int\int_{\partial A^+} \vec{F} \cdot d\vec{S} = \pi.$$

[4pts] (c) Calculate $\int\int_{\partial A^+} \vec{F} \cdot d\vec{S}$ indirectly using Gauss' Divergence Theorem.