MAT 2122, Fall 2019 – Midterm exam 3 **Instructor: A. Tikuisis**

Solutions

Read the following instructions:

- The use of cellphones, electronic devices (including calculators), and course notes is strictly forbidden. All phones and electronic devices must be turned off and kept in your bags: do not leave them on you. If you are seen to have an electronic device on your person, we may ask you to leave the exam immediately, and fraud allegations could be made, which could lead to a mark of 0 (zero) on this midterm.
- The duration of this midterm is 75 minutes.
- This is a closed book midterm containing **5 questions**.
- There is an additional blank page at the end of this exam that you may use as scrap paper. If you run out of space, you may use this page or the backs of pages. Clearly indicate where to find your answer.
- Do not detach the pages of this test, apart from the last (blank) page. If you detach the last page, do not use it for your submitted answers.
- You must give clear and complete solutions, with calculations, explanations and justifications. Make sure that your answer is clearly indicated; you must convince me that you understand your solution in order to receive full marks.

By signing below, you acknowledge that you are required to respect the above statements.

Signature: ______________________________

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	Total
Mark						
Out of	9	12	6	12	11	50

1. Multiple choice. Write your answer clearly in the blank below the question, or write “X” to indicate blank. Each question is worth **3 marks** and has exactly one correct answer. A correct solution is worth 3 marks, an incorrect or blank solution is worth 0 marks, and “X” (intentional blank) is worth 1 mark.

(i) Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field and suppose that

$$D\vec{F}(0,0,0) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

What is $\mathrm{curl}(\vec{F})(0,0,0)$?

(A) $(1,2,2)$.

(B) 5.

(C) $(2,1,2)$.

(D) 1.

(E) $(1,1,0)$.

Solution: <u>(B) (1,1,0)</u>

$$\begin{aligned} \mathrm{curl}(\vec{F}) &= (\frac{\partial F_2}{\partial y} - \frac{\partial F_3}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}) \\ &= (1-0, 1-0, 0-0). \end{aligned}$$

(ii) Let $\Phi : [-1,1] \times [-1,1] \to \mathbb{R}^3$ be a C^1 parametrized surface and suppose that

$$\Phi(0,0) = (1,1,1), \quad D\Phi(0,0) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Which of the following points is on the tangent plane to the surface at $(1,1,1)$?

(A) $(4,7,0)$.

(B) $(6,2,3)$.

(C) $(0,0,0)$.

(D) $(1,2,1)$.

(E) $(2,1,1)$.

Solution: <u>(A) (4,7,0)</u>

A normal vector is $\vec{n} := (1,2,1) \times (0,0,1) = (2,-1,0)$. Hence an equation is $0 = \vec{n} \cdot \big((x,y,z) - (1,1,1)\big) = 2(x-1) - (y-1) = 2x - y - 1$. From the given list, $(4,7,0)$ is the only one satisfying this equation.

Alternatively, the parametric form of the tangent plane is $p(s,t) = (1,1,1) + s(1,2,1) + t(0,0,1)$. The point $(4,7,0)$ can be written as $p(3,-4) = (1,1,1) + 3(1,2,1) - 4(0,0,1)$.

(iii) Let $p, q : [0,1] \to \mathbb{R}^2$ be paths defined by

$$p(t) := (t,t), \quad q(t) := (1-t, 1-t).$$

Suppose that $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field and

$$\int_p 1\, ds = 2, \quad \int_p \vec{F} \cdot d\vec{s} = 5.$$

Which of the following is correct?

(A) $\int_q 1\, ds = 2$ and $\int_q \vec{F} \cdot d\vec{s} = 5$.

(B) $\int_q 1\, ds = -2$ and $\int_q \vec{F} \cdot d\vec{s} = -5$.

(C) $\int_q 1\, ds = 2$ and $\int_q \vec{F} \cdot d\vec{s} = -5$.

(D) $\int_q 1\, ds = -2$ and $\int_q \vec{F} \cdot d\vec{s} = 5$.

(E) There is not enough information to compute $\int_q 1\, ds$ and $\int_q \vec{F} \cdot d\vec{s}$.

Solution: (C) 2,-5.

The two paths parametrize the same curve, with opposite orientations. Hence $\int_q 1\, ds = \int_p 1\, ds$ and $\int_q \vec{F} \cdot d\vec{s} = -\int_p \vec{F} \cdot d\vec{s}$.

There is a mistake on this question: the path length $\int_p 1\, ds$ is in fact $\sqrt{2}$, not 2. This means $\int_q 1\, ds = \sqrt{2}$ as well.

2. Let S be a surface parametrized by $\Phi(u,v) := (u^2/2, v^2, -uv)$ where $u \in [0,1]$ and $v \in [0,1]$.

(i) Determine $\Phi_u \times \Phi_v$ and show that $\|\Phi_u \times \Phi_v\| = u^2 + 2v^2$. 7

Solution: $\Phi_u = (u, 0, -v)$ and $\Phi_v = (0, 2v, -u)$. 2

Thus $\Phi_u \times \Phi_v = (u, 0, -v)x(0, 2v, -u) = (2v^2, u^2, 2uv)$. 2

Also, $\|\Phi_u \times \Phi_v\| = \sqrt{(-2v^2)^2 + (u^2)^2 + (2uv)^2}$ 2

$$= \sqrt{4v^4 + 4u^2v^2 + u^4}$$

$$= \sqrt{(2v^2 + u^2)^2}$$

$$= 2v^2 + u^2.$$ 1

(ii) Compute the surface area of S. 5

Solution: Surface area $= \int_0^1 \int_0^1 \|\Phi_u \times \Phi_v\| \, du \, dv$ 2

$$= \int_0^1 \int_0^1 2v^2 + u^2 \, du \, dv$$ 2

$$= \int_0^1 2v^2 + \frac{1}{3} \, dv$$

$$= \frac{2}{3} + \frac{1}{3} = 1$$ 1

3. **(i)** If $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a differentiable vector field, define $\mathrm{div}(\vec{F})$.

ANSWER: $\mathrm{div}(\vec{F}) = \nabla \cdot F$ or $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ where $F = (F_1, F_2, F_3)$. 3

(ii) Define the vector field $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\vec{G}(x, y, z) := (x + y, x^2 + z, yz).$$

Compute $\mathrm{div}(\vec{G})$. 3

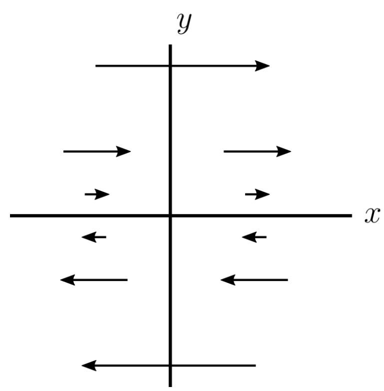
Solution: We have $\frac{\partial G_1}{\partial x} = 1$, $\frac{\partial G_2}{\partial y} = 0$, and $\frac{\partial G_3}{\partial z} = y$. 2

Hence, $\mathrm{div}(\vec{G}) = 1 + 0 + y$. 1

4. (i) Let $q : [a, b] \to \mathbb{R}^2$ be a path. We may rewrite $\int_q ye^x\,dx + x\ln(1+y^2)\,dy$ as $\int_q \vec{F} \cdot d\vec{s}$ for some vector field $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$. What is this vector field?

ANSWER: $\vec{F}(x, y) = (ye^x, x\ln(1+y^2))$. 3

(ii) Sketch the vector field $\vec{G}(x, y) := (y, 0)$ below. 3



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(iii) Define the path $p(t) := (t, t^2, t^3)$ for $t \in [0, 1]$, and define the vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\vec{F}(x, y, z) = (x, z - x^3, y - x^2).$$

Compute $\int_p \vec{F} \cdot d\vec{s}$. 6

Solution: We have $p'(t) = (1, 2t, 3t^2)$ 2

and $\vec{F}(p(t)) = (t, t^3 - t^3, t^2 - t^2) = (t, 0, 0)$. 2

Therefore $\vec{F}(p(t)) \cdot p'(t) = (1, 2t, 3t^2) \cdot (t, 0, 0) = t + 0 + 0 = t$. 1

Hence,

$\int_p \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F}(p(t)) \cdot p'(t)\,dt$

$= \int_0^1 t\,dt = \frac{1}{2}$. 1

5. Let A be the cube $[0,1] \times [0,1] \times [0,1]$ and suppose that this object has a density function given by $\delta(x,y,z) := 2x + 3y^2$.

(i) Prove that the mass of the cube is 2. 4

Solution:

$$\text{Mass} = \int\int\int_A \delta(x,y,z)\,dx\,dy\,dz \qquad 2$$
$$= \int_0^1\int_0^1\int_0^1 2x + 3y^2\,dx\,dy\,dz \qquad 1$$
$$= \int_0^1\int_0^1 1 + 3y^2\,dy\,dz$$
$$= \int_0^1 1 + 1\,dy\,dz \qquad 1$$
$$= 2.$$

(ii) Determine the centre of mass of this cube. 7

Solution: The centre of mass is (x_0, y_0, z_0) where

$$x_0 = \frac{1}{\text{mass}}\int\int\int_A x\delta(x,y,z)\,dx\,dy\,dz, \qquad 2$$
$$y_0 = \frac{1}{\text{mass}}\int\int\int_A y\delta(x,y,z)\,dx\,dy\,dz,$$
$$z_0 = \frac{1}{\text{mass}}\int\int\int_A z\delta(x,y,z)\,dx\,dy\,dz.$$

So we compute

$$x_0 = \tfrac{1}{2}\int_0^1\int_0^1\int_0^1 x(2x+3y^2)\,dx\,dy\,dz \qquad 2$$
$$= \tfrac{1}{2}\int_0^1\int_0^1\int_0^1 2x^2 + 3y^2x\,dx\,dy\,dz$$
$$= \tfrac{1}{2}\int_0^1\int_0^1 \tfrac{2}{3} + \tfrac{3y^2}{2}\,dy\,dz$$
$$= \tfrac{1}{2}\int_0^1 \tfrac{2}{3} + \tfrac{1}{2}\,dz$$
$$= \tfrac{1}{2}\left(\tfrac{2}{3} + \tfrac{1}{2}\right) \qquad 1$$
$$= \tfrac{7}{12}.$$

(The last step is not necessary).

$$y_0 = \tfrac{1}{2}\int_0^1\int_0^1\int_0^1 y(2x+3y^2)\,dx\,dy\,dz \qquad 1$$
$$= \tfrac{1}{2}\int_0^1\int_0^1\int_0^1 2xy + 3y^3\,dx\,dy\,dz$$
$$= \tfrac{1}{2}\int_0^1\int_0^1 y + 3y^3\,dy\,dz$$
$$= \tfrac{1}{2}\int_0^1 \tfrac{1}{2} + \tfrac{3}{4}\,dz$$
$$= \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{3}{4}\right) \qquad 1$$
$$= \tfrac{5}{8}.$$

(The last step is not necessary).

$$z_0 = \tfrac{1}{2}\int_0^1\int_0^1\int_0^1 z(2x+3y^2)\,dx\,dy\,dz$$ 1

$$= \tfrac{1}{2}\int_0^1\int_0^1 z(1+3y^2)\,dy\,dz$$

$$= \tfrac{1}{2}\int_0^1 2z\,dz$$

$$= \tfrac{1}{2}.$$ 1

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