MAT 2122, Fall 2019 – Midterm exam 2 **Instructor: A. Tikuisis**

Solutions

Read the following instructions:

- The use of cellphones, electronic devices (including calculators), and course notes is strictly forbidden. All phones and electronic devices must be turned off and kept in your bags: do not leave them on you. If you are seen to have an electronic device on your person, we may ask you to leave the exam immediately, and fraud allegations could be made, which could lead to a mark of 0 (zero) on this midterm.
- The duration of this midterm is 75 minutes.
- This is a closed book midterm containing **5 questions**.
- There is an additional blank page at the end of this exam that you may use as scrap paper. If you run out of space, you may use this page or the backs of pages. Clearly indicate where to find your answer.
- Do not detach the pages of this test, apart from the last (blank) page. If you detach the last page, do not use it for your submitted answers.
- You must give clear and complete solutions, with calculations, explanations and justifications. Make sure that your answer is clearly indicated; you must convince me that you understand your solution in order to receive full marks.

By signing below, you acknowledge that you are required to respect the above statements.

Signature: ____________________________

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	Total
Mark					
Out of	9	11	15	15	50

1. Multiple choice. Write your answer clearly in the blank below the question, or write "X" to indicate blank. Each question is worth **3 marks** and has exactly one correct answer. A correct solution is worth 3 marks, an incorrect or blank solution is worth 0 marks, and "X" (intentional blank) is worth 1 mark.

(i) Let

$$X := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\},$$
$$Y := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\},$$
$$Z := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

and let $f : X \to \mathbb{R}$ be a C^1 function. If $\nabla f(x, y, z) \neq (0, 0, 0)$ for all $(x, y, z) \in Y$ then:

(A) f does not attain a global maximum.

(B) f attains a global maximum at some point $(x, y, z) \in Z$.

(C) $\nabla f(x, y, z) = (0, 0, 0)$ for some $(x, y, z) \in Z$.

(D) Both (B) and (C).

(E) This is impossible; such a function f cannot exist.

Solution: <u>(B)</u>.
Since f is continuous and the set X is bounded and contains its boundary, it must attain a maximum. By the First Derivative Test, it doesn't attain its maximum on Y, so it follows that it attains its maximum on Z.

Since Z is the boundary of X, the First Derivative Test doesn't apply, so we might not have $\nabla f(x, y, z) = (0, 0, 0)$ where f attains its maximum (or minimum).

(ii) Let $R := [a, b] \times [c, d]$ and $f : R \to \mathbb{R}$ be a function. Under which conditions is $\int\int_R f(x, y)\, dx\, dy$ defined?

(A) If f is a bounded continuous function.

(B) (A) and more generally, if f is a bounded function and the set of points where f is not continuous is contained in a finite union of graphs of continuous functions, and $\int_a^b f(x, y)\, dx$ exists for each $y \in [c, d]$ and $\int_c^d f(x, y)\, dy$ exists for each $x \in [a, b]$.

(C) (B) and more generally, if f is a bounded function and the set of points where f is not continuous is contained in a finite union of graphs of continuous functions.

(D) (C) and more generally, if f is any bounded function.

(E) (D) and more generally, for any function f.

Solution: (C).

(iii) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) := (x - y, 2x - 5y)$, and suppose $P, Q \subseteq \mathbb{R}^2$ are parallellograms satisfying $T(Q) = P$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that

$$\int_P f\, dA = 3.$$

What is $\int_Q (f \circ T)\, dA$?

(A) 1.

(B) -1.

(C) 3.

(D) 9.

(E) -9.

Solution: (A)

We have

$$DT(x,y) = \begin{bmatrix} 1 & -1 \\ 2 & -5 \end{bmatrix},$$

and so $|\det DT(x,y)| = |-5+2| = 3$. Thus we have

$$3 = \int_P f\,dA = \int_Q (f \circ T)|\det JT|\,dA = 3\int_Q f \circ T\,dA,$$

which implies that $\int_Q f \circ T\,dA = 1$.

2. Consider the following integral:

$$\int_1^e \int_1^x \frac{1}{y}\, dy\, dx.$$

(i) What region in $\mathbb{R}^2$ is this integral taken over? That is, if this integral is

$$\int \int_D \frac{1}{y}\, dy\, dx$$

then what is the set D? 3

Solution: $D = \{(x,y) : x \in [1,e],\ y \in [1,x]\}$. 3

A correct picture (only) earns 2.

(ii) Evaluate the integral. (*Hint. Change the order of integration.*) 8

Solution:

$\int_1^e \int_1^x \frac{1}{y}\, dy\, dx = \int_1^e \int_y^e \frac{1}{y}\, dx\, dy$ 2

$= \int_1^e \frac{e}{y} - 1\, dx\, dy$ 2

$= e\log(y) - y|_{y=1}^e$ 2

$= e\log(e) - e - (e\log(1) - 1).$ 2

$= e - e - (0-1) = 1.$

(The last step is not necessary.)

Alternate solution:

$\int_1^e \int_1^x \frac{1}{y}\, dy\, dx = \int_1^e \log(y)|_{y=1}^x\ dx$ 2

$= \int_1^e \log(x)\, dx$ 2

$= x\log(x) - x|_{x=1}^e$ 2

$= e\log(e) - e - (1\log(1) - 1).$ 2

3. Use the Lagrange multiplier method to show that the maximum of

$$f(x,y,z) := xy + z \quad \text{on} \quad S := \{(x,y,z) : x^2 + y^2 + z^2 = 9\}$$

occurs at the points $(2,2,1)$ and $(-2,-2,1)$, and at no other points. 15

Solution: $\nabla f = (y, x, 1)$, $\nabla g = (2x, 2y, 2z)$. 2

The system of equations to solve is

(1) $y = 2\lambda x$,

(2) $x = 2\lambda y$,

(3) $1 = 2\lambda z$,

(4) $x^2 + y^2 + z^2 = 9$. 3

Multiply (1) by x and (2) by y to get

$$2\lambda x^2 = xy = 2\lambda y^2.$$

Then by (3), $\lambda \neq 0$ and so $x = \pm y$. 2

If $x = y \neq 0$ then $x = 2\lambda x$ and so $\lambda = \frac{1}{2}$ which implies that $z = 1$. In this case, we have $2x^2 + 1 = 9$, so that $x = \pm 2$. We obtain the points $(2,2,1)$ and $(-2,-2,1)$, and at both of those points, $f = 5$. 2

Likewise, if $x = -y$ then $x = -2\lambda x$ and so $\lambda = -\frac{1}{2}$ which implies that $z = -1$. In this case again we have $2x^2 + 1 = 9$, so that $x = \pm 2$. We obtain the points $(-2,2,-1)$ and $(2,-2,-1)$, and at both of those points, $f = -5$. 2

Finally, if $x = y = 0$ then from $x^2 + y^2 + z^2 = 9$ we get $z = \pm 3$. This gives the points $(0,0,3)$, where $f = 3$, and $(0,0,-3)$, where $f = -3$. 2

Hence the maximum is 5 and it is attained at the two points $(2,2,1)$ and $(-2,-2,1)$. 2

4. Let $B := \{(x, y) : x, y \geq 0, x^2 + y^2 \leq 1\}$.

(i) Write down a set $A \subseteq \mathbb{R}^2$ such that

$$B = \{(r\cos(\theta), r\sin(\theta)) : (r, \theta) \in A\}.$$ 5

Solution: $r^2 = x^2 + y^2 \leq 1$, so we ask that $r \in [0, 1]$. 1

The condition $x, y \geq 0$ means that we are in the first quadrant, so we ask that $\theta \in [0, \pi/2]$. 2

Thus, $A = [0, 1] \times [0, \frac{\pi}{2}]$. 2

(ii) Use polar coordinates to evaluate

$$\int\int_B xy\, dx\, dy$$

(*The double-angle identity,* $\sin(t)\cos(t) = \frac{1}{2}\sin(2t)$, *may be useful.*) 10

Solution: Using A from part (i) and the polar coordinates change-of-variable formula, we have

$\int\int_B xy\, dx\, dy = \int\int_A r\cos(\theta) r\sin(\theta) r\, d\theta\, dr$ 2

$= \int_0^1 \int_0^{\pi/2} r\cos(\theta) r\sin(\theta) r\, d\theta\, dr$ 1

$= \int_0^1 r^3\, dr \int_0^{\pi/2} \frac{1}{2}\sin(2\theta)\, d\theta$ 3

$= \frac{r^4}{4}\Big|_{r=0}^1 \Big(-\frac{1}{4}\cos(2\theta)\Big|_{\theta=0}^{\pi/2}\Big)$ 3

$= -\frac{1}{4} \cdot \frac{1}{4}(-2) = \frac{1}{8}.$ 1

(the last page)