MAT 2122, Fall 2019 – Midterm Exam 1 **Instructor: A. Tikuisis**

Last name:	______________________
First name:	______________________
Student number:	______________________

Read the following instructions:

- The use of cellphones, electronic devices (including calculators), and course notes is strictly forbidden. All phones and electronic devices must be turned off and kept in your bags: do not leave them on you. If you are seen to have an electronic device on your person, we may ask you to leave the exam immediately, and fraud allegations could be made, which could lead to a mark of 0 (zero) on this midterm.
- The duration of this midterm is 75 minutes.
- This is a closed book midterm containing **5 questions**.
- Do not detach the pages of this test.
- There is an additional blank page at the end of this exam that you may use as scrap paper. If you run out of space, you may use this page or the back of pages. Clearly indicate where to find your answer if it is not entirely contained in the space provided on the page.
- You must give clear and complete solutions, with calculations, explanations and justifications. Make sure that your answer is clearly indicated; you must convince me that you understand your solution in order to receive full marks.

By signing below, you acknowledge that you are required to respect the above statements.

Signature: ______________________

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	Total
Mark						
Out of	9	10	10	10	11	50

1. Multiple choice. Please circle your answer, or in case you change your mind, write your answer clearly below the question. Each question is worth **3 marks** and has exactly one correct answer. A correct solution is worth 3 marks, an incorrect solution is worth 0 marks, and no solution is worth 1 mark.

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function, and suppose that

$$\frac{\partial f}{\partial x}(0,0), \quad \frac{\partial f}{\partial y}(0,0)$$

exist. Which of the following is correct:

(i) If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $(0,0)$ then f is differentiable at $(0,0)$.

(ii) If f is differentiable at $(0,0)$ then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $(0,0)$.

(iii) If $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0)$ then f is differentiable at $(0,0)$.

(iv) If f is differentiable at $(0,0)$ then $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0)$.

(v) None of the above.

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function, and suppose that

$$\lim_{x\to 0} f(x,0,0) = 0 \quad \text{and} \quad \lim_{y\to 0} f(0,y,0) = 0.$$

Which of the following is correct:

(i) If $\lim_{z\to 0} f(0,0,z) = 0$ then $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = 0$.

(ii) If $\lim_{z\to 0} f(0,0,z) = c$ then $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = (0,0,c)$.

(iii) If $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = 0$ then $\lim_{z\to 0} f(0,0,z) = 0$.

(iv) If $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$ doesn't exist, then $\lim_{z\to 0} f(0,0,z)$ doesn't exist.

(v) None of the above.

(c) Suppose a particle is moving along the surface $z = f(x, y)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function. The (x, y)-coordinates of the particle are given by the differentiable function $c : \mathbb{R} \to \mathbb{R}^2$. Let $t_0 \in \mathbb{R}$, and suppose

$$c(t_0) = (1, 3), \quad c'(t_0) = (-1, 4), \quad \nabla f(1, 3) = (3, 1), \quad \text{and} \quad \nabla f(-1, 4) = (0, 1).$$

What is the velocity of the particle at time t_0?

(i) $(1, 3, 1)$.

(ii) $(1, 3, 4)$.

(iii) $(1, 3, 6)$.

(iv) $(-1, 4, 1)$.

(v) $(-1, 4, 4)$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := xe^y + e^x$.

(a) Compute the gradient $\nabla f(x, y)$. (3)

(b) Compute $\frac{\partial^2 f}{\partial x \partial y}$. (3)

(c) Give a formula for the tangent plane of f at $(0, 0)$. (4)

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function, and suppose we know the following.

$$f(0,0) = 3, \quad \nabla f(0,0) = (7,2),$$
$$Hf(0,0) = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}.$$

(a) Write out the 2^{nd}-order Taylor polynomial for f centred at $(0,0)$. (5)

(b) Define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) := f(x,y) - 7x - 2y.$$

Prove that $(0,0)$ is a critical point of g, and determine whether $(0,0)$ is a local maximum, a local minimum, or a saddle point. (5)

4. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function satisfying

$$f(x, y) = -f(y, x) \text{ for all } (x, y) \in \mathbb{R}^2.$$

(a) Prove that for $(a, b) \in \mathbb{R}^2$,

$$\frac{\partial f}{\partial x}(a, b) = -\frac{\partial f}{\partial y}(b, a).$$ (5)

(b) Let S be the level set of f at $k = 0$. Suppose that

$$(1, 2) \in S, \quad \frac{\partial f}{\partial x}(1, 2) = 0, \quad \text{and} \quad \frac{\partial f}{\partial y}(1, 2) \neq 0.$$

Prove that $(2, 1) \in S$ and that the tangent line of S at $(1, 2)$ is orthogonal to the tangent line of S at $(2, 1)$. (5)

5. (a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be given by $f(x) = (f_1(x), \ldots, f_m(x))$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Define $Df(a)$ where $a \in \mathbb{R}^n$.

ANSWER: $Df(a) =$ (2)

(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ are both differentiable functions. According to the Chain Rule, what is a formula for $D(g \circ f)(a)$, where $a \in \mathbb{R}^n$? (No justification is required.)

ANSWER: $D(g \circ f)(a) =$ (4)

(c) Let $c : \mathbb{R} \to \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions, with $c(t) = (c_1(t), c_2(t))$. Using the Chain Rule multiple times, show that

$$(f \circ c)''(t) = \frac{\partial^2 f}{\partial x^2}(c(t))\,(c_1'(t))^2 + 2\frac{\partial^2 f}{\partial x \partial y}(c(t))\,c_1'(t)c_2'(t) + \frac{\partial^2 f}{\partial y^2}(c(t))\,(c_2'(t))^2.$$

(5)

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