MAT 2122, Fall 2019 – Final exam   Instructor: A. Tikuisis

Last name:	______________________
First name:	______________________
Student number:	______________________

Read carefully:

- Cellular phones, electronic devices (including calculators) or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: academic fraud allegations will be filed which may result in your obtaining a *0* (zero) for the exam.
- This is a closed book exam containing **5 questions**.
- There are two additional blank pages at the end of this exam that you may use as scrap paper. If you run out of space, you may use this page or the backs of pages. Clearly indicate where to find your answer.
- Do not detach the pages of this test, apart from the last (blank) page. If you detach the last page, **do not** use it for your submitted answers.
- You must give clear and complete solutions, with calculations, explanations and justifications. Make sure that your answer is clearly indicated; you must convince me that you understand your solution in order to receive full marks.

By signing below, you acknowledge that you are required to respect the above statements.

Signature: ______________________

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	Total
Mark						
Out of	21	17	15	30	17	100

1. Multiple choice. Use the following table to record your answers. Write "A", "B", "C", "D", or "E" to indicate that you have chosen that response, or write "X" to indicate blank (no response). A correct solution is worth `3 marks`, an incorrect or blank solution is worth `0 marks`, and "X" (intentional blank) is worth `1 mark`.

Question part	**(i)**	**(ii)**	**(iii)**	**(iv)**	**(v)**	**(vi)**	**(vii)**
Response							

(i) Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) := x + x^2 + 2xy + ye^z$, and let $\vec{u} := (\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$. What is the directional derivative of f at $(0, 0, 0)$ in the direction of $\vec{u}$?

(A) $\left(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$.

(B) 1.

(C) $\frac{7}{3}$.

(D) $\left(\frac{2}{3}, \frac{1}{3}, 0\right)$.

(E) $(1, 1, 0)$.

(ii) Suppose that S is the level set of a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$ and that $C \subseteq \mathbb{R}^2$ is a curve parametrized by the C^1 function $p : [-1, 1] \to \mathbb{R}^2$. Suppose also that $p(0) = (0, 0)$, $f(0, 0) = 0$, and $p'(0) = \nabla f(0, 0) \neq \vec{0}$. Which of the following is true?

(A) $(0, 0) \in C$ and $(0, 0) \notin S$.

(B) $(0, 0) \in S$ and $(0, 0) \notin C$.

(C) $(0, 0) \in S \cap C$ and the tangent line to S at $(0, 0)$ is orthogonal to the tangent line to C at $(0, 0)$.

(D) $(0, 0) \in S \cap C$ and the tangent line to S at $(0, 0)$ is the same as the tangent line to C at $(0, 0)$.

(E) $(0, 0) \in S \cap C$ but the tangent line to S and/or C at $(0, 0)$ might not exist.

(iii) Let $A := \{(x,y,z) : x^2 + y^2 \leq z^2, z \in [0,1], x \geq 0, y \geq 0\}$. What is $\int\int\int_A z(x^2+y^2)\,dx\,dy\,dz$?

(A) $\frac{\pi}{12}$.

(B) $\frac{4}{3}$.

(C) $\frac{4}{9}$.

(D) $\frac{\pi}{6}$.

(E) $\frac{\pi}{4}$.

(iv) Consider a wire that is parametrized by the path $c : [0,1] \to \mathbb{R}^3$ given by

$$c(t) := (t, e^{2t}, 2e^t),$$

and with density function given by $\delta(x,y,z) := \frac{2}{1+2y}$. What is the centre of mass of this wire?

(A) $\big(\frac{1}{2}, \frac{e^2}{2}, e\big)$.

(B) $\big(\frac{1}{2}, \frac{e^2-1}{2}, 2(e-1)\big)$.

(C) $\big(\frac{1}{2}, \frac{e^2}{2}, \frac{e}{2}\big)$.

(D) $\big(\frac{1}{e^2} + \frac{1}{2}, \frac{1+e^2}{2} - \frac{1}{e^2}, \frac{2}{e} + \frac{4e}{3} - \frac{4}{3e^2}\big)$.

(E) $\big(1 + \frac{e^2}{2}, \frac{e^2+e^4}{2} - 1, 2e + \frac{4e^3}{3} - \frac{5}{3}\big)$.

(v) For each $R > 0$, define $T_R : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ by

$$T_R(\phi, \theta) := (R\cos(\theta)\sin(\phi), R\sin(\theta)\sin(\phi), R\cos(\phi)).$$

Note that T_R parametrizes the sphere of radius R centred at the origin. Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and suppose that

$$\int\int_{T_R} \vec{F} \cdot d\vec{S} = \sqrt{R}.$$

Set $A := \{(x, y, z) : x^2 + y^2 + z^2 \leq 16\}$. What is

$$\int\int\int_A \mathrm{div}\vec{F}\, dx\, dy\, dz?$$

(A) $\frac{16}{3}$.

(B) 2.

(C) $\frac{1}{4}$.

(D) 4.

(E) There is not enough information to determine the answer.

(vi) Let $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$\vec{F}(x, y) := \left(y\cos(xy)^2 + \frac{y}{1 + e^x}, x\cos(xy)^2 + x + \log(e^x + 1)\right).$$

Which of the following is true? Select the most complete answer.

(A) $\vec{F} = \nabla f$ for some function $f : \mathbb{R}^2 \to \mathbb{R}$.

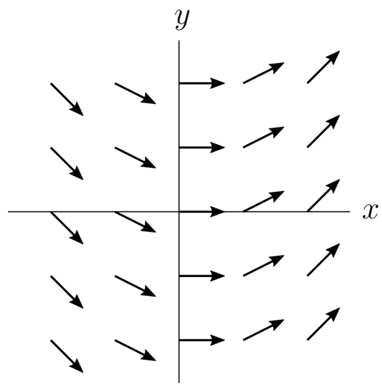
(B) $\vec{F}$ is a C^1 function.

(C) If $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field defined by $\vec{G}(x, y, z) := (\vec{F}(x, y), 0)$, then $\mathrm{curl}\,\vec{G} = (0, 0, 0)$.

(D) Both (ii) and (iii).

(E) All of the above.

(vii) Which vector field is shown in the following sketch?



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(A) $\vec{F}(x, y) = \left(\frac{1}{\sqrt{1+y^2}}, \frac{y}{\sqrt{1+y^2}}\right)$.

(B) $\vec{F}(x, y) = (1, x)$.

(C) $\vec{F}(x, y) = (1, y)$.

(D) $\vec{F}(x, y) = \left(\frac{1}{\sqrt{1+x^2}}, \frac{x}{\sqrt{1+x^2}}\right)$.

(E) None of the above.

2. Define $A := \{(x,y) : x^2 + y^2 \leq 1, y \geq x\}$ and define $f : A \to \mathbb{R}$ by

$$f(x,y) := xy^3 - y^3.$$

(i) Does f attain a maximum and a minimum on A? Explain why. 3

(ii) Find all local and global maxima and minima of f an A. 14

3. Define the parametrized surface $\Phi : [0,1] \times [0,1] \to \mathbb{R}^3$ by

$$\Phi(u,v) := (e^u + e^{-v}, e^u + e^{-v}, u + v).$$

Find the average value of $f(x,y) := \frac{1}{y}$ over $\Phi([0,1] \times [0,1])$. 15

4. Let $D := [0,1] \times [0,1]$, let $\Phi : D \to \mathbb{R}^3$ be the parametrized surface given by

$$\Phi(u,v) := (u, v, u(u-1)v(v-1)),$$

and define the vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\vec{F}(x,y,z) = (2xye^{y^2}, xye^z - e^{y^2}, -xe^z).$$

(i) Determine the boundary of $\Phi(D)$ in the sense of a surface, that is, the set $\partial(\Phi(D))$ as in Stokes' Theorem. 5

(ii) Determine $\Phi_u \times \Phi_v$. 5

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(iii) Write out $\int\int_\Phi \vec{F} \cdot d\vec{S}$ as a double-integral over D. That is, write it as $\int\int_D f(u,v)\,du\,dv$ for a function f. You should fully expand the function f (i.e., terms like Φ and $\vec{F}$ should not appear in your final answer). **Do not evaluate this integral**. 5

(iv) Define the vector field $\vec{G} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\vec{G}(x,y,z) := (xye^z, 0, xe^{y^2})$. Show that $\operatorname{curl} \vec{G} = \vec{F}$. 5

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(v) Evaluate $\int\int_{\Phi} \vec{F} \cdot d\vec{S}$ by using Stokes' Theorem. 10

5. Let $\vec{F} = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field and let $\Phi = (\Phi_1, \Phi_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function. Define $\vec{G} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned}\vec{G} &:= ((\vec{F}\circ\Phi)\cdot\Phi_v, -(\vec{F}\circ\Phi)\cdot\Phi_u) \\ &= ((F_1\circ\Phi)\frac{\partial\Phi_1}{\partial v} + (F_2\circ\Phi)\frac{\partial\Phi_2}{\partial v}, -(F_1\circ\Phi)\frac{\partial\Phi_1}{\partial u} - (F_2\circ\Phi)\frac{\partial\Phi_2}{\partial u}).\end{aligned}$$

Here is a calculation of $\mathrm{div}(\vec{G})$.

$$\begin{aligned}
\mathrm{div}(\vec{G}) &= \frac{\partial}{\partial u}\big((F_1\circ\Phi)\frac{\partial\Phi_1}{\partial v} + (F_2\circ\Phi)\frac{\partial\Phi_2}{\partial v}\big) \\
&\quad + \frac{\partial}{\partial v}\big(-(F_1\circ\Phi)\frac{\partial\Phi_1}{\partial u} - (F_2\circ\Phi)\frac{\partial\Phi_2}{\partial u})\big) && (1)\\
&= \frac{\partial(F_1\circ\Phi)}{\partial u}\frac{\partial\Phi_1}{\partial v} + (F_1\circ\Phi)\frac{\partial^2\Phi_1}{\partial u\partial v} + \frac{\partial(F_2\circ\Phi)}{\partial u}\frac{\partial\Phi_2}{\partial v} + (F_2\circ\Phi)\frac{\partial^2\Phi_2}{\partial u\partial v} \\
&\quad - \frac{\partial(F_1\circ\Phi)}{\partial v}\frac{\partial\Phi_1}{\partial u} - (F_1\circ\Phi)\frac{\partial^2\Phi_1}{\partial v\partial u} - \frac{\partial(F_2\circ\Phi)}{\partial v}\frac{\partial\Phi_2}{\partial u} - (F_2\circ\Phi)\frac{\partial^2\Phi_2}{\partial v\partial u} && (2)\\
&= \frac{\partial(F_1\circ\Phi)}{\partial u}\frac{\partial\Phi_1}{\partial v} + \frac{\partial(F_2\circ\Phi)}{\partial u}\frac{\partial\Phi_2}{\partial v} - \frac{\partial(F_1\circ\Phi)}{\partial v}\frac{\partial\Phi_1}{\partial u} - \frac{\partial(F_2\circ\Phi)}{\partial v}\frac{\partial\Phi_2}{\partial u} && (3)\\
&= \big(\frac{\partial F_1}{\partial x}\circ\Phi\big)\frac{\partial\Phi_1}{\partial u}\frac{\partial\Phi_1}{\partial v} + \big(\frac{\partial F_1}{\partial y}\circ\Phi\big)\frac{\partial\Phi_2}{\partial u}\frac{\partial\Phi_1}{\partial v} \\
&\quad + \big(\frac{\partial F_2}{\partial x}\circ\Phi\big)\frac{\partial\Phi_1}{\partial u}\frac{\partial\Phi_2}{\partial v} + \big(\frac{\partial F_2}{\partial y}\circ\Phi\big)\frac{\partial\Phi_2}{\partial u}\frac{\partial\Phi_2}{\partial v} \\
&\quad - \big(\frac{\partial F_1}{\partial x}\circ\Phi\big)\frac{\partial\Phi_1}{\partial v}\frac{\partial\Phi_1}{\partial u} - \big(\frac{\partial F_1}{\partial y}\circ\Phi\big)\frac{\partial\Phi_2}{\partial v}\frac{\partial\Phi_1}{\partial u} \\
&\quad - \big(\frac{\partial F_2}{\partial x}\circ\Phi\big)\frac{\partial\Phi_1}{\partial v}\frac{\partial\Phi_2}{\partial u} - \big(\frac{\partial F_2}{\partial y}\circ\Phi\big)\frac{\partial\Phi_2}{\partial v}\frac{\partial\Phi_2}{\partial u} && (4)\\
&= \big(\frac{\partial F_1}{\partial y}\circ\Phi - \frac{\partial F_2}{\partial x}\circ\Phi\big)\big(\frac{\partial\Phi_2}{\partial u}\frac{\partial\Phi_1}{\partial v} - \frac{\partial\Phi_1}{\partial u}\frac{\partial\Phi_2}{\partial v}\big). && (5)
\end{aligned}$$

In the above proof, which of the following are used and where? Write the line number ((1)–(5)) beside each one that is used, and leave the rest blank. 17

Chain Rule ____

The Change-of-Variable Theorem ____

Equality of mixed second order partial derivatives ____

Fubini's Theorem ____

Gauss' Divergence Theorem ____

Lagrange Multiplier Theorem ____

Green's Theorem ____

Mean Value Theorem ____

The Product Rule for differentiation ____

Stokes' Theorem ____

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