Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Mathematical Reasoning and Proofs MAT1362

Second Midterm Exam (α)

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Instructions. *You must sign below to confirm that you have read, understand, and will follow them.*

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 5 questions on 9 pages. Page 9 contains additional work space. *Please do not detach it.*
- Be sure to read carefully and follow the instructions for the individual problems. To receive full marks, your solution/proof must be correct, complete, and show all relevant details.
- For rough work or additional work space, you may use the back pages. Do not use scrap paper of your own.
- Use correct mathematical notation and terminology, as defined in class.
- You may ask for clarification.
- Unauthorized electronic devices (such as cellular phones) are not permitted during this exam. Such devices must be *turned off completely* and stored out of students' reach. Students found in possession of such a device during the exam will be asked to leave immediately and academic fraud allegations may be filed.

LAST NAME:____________________________

First name:____________________________

Signature:____________________________

Write your student number on the next page.

Circle your DGD (this is where you will pick up your marked exam):

D01	**D02**
11:30	13:00
MRT 015	MRT 219

Student number:______________

Question	1	2	3	4	5	Total
Max	7	7	7	7	7	35
Marks						

[7pts] **(1)** Define an integer sequence $(a_n)_{n=1}^{\infty}$ recursively as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 4 \\ a_n &= 3a_{n-1} + 4a_{n-2} \qquad \text{for } n \geq 3 \end{aligned}$$

(a) Use the recurrence to evaluate a_3, a_4, and a_5. Then express each of the terms $a_1, a_2, \ldots, a_5$ as a power of 2.

(b) Using (a), guess an explicit formula for a_n; that is, give an expression for a_n in terms of n only (without using recursion).

(c) Use Strong Induction to prove your formula from (b).
Clearly state the proposition to be proved, the Basis of Induction, the Induction Step, and the Induction Hypothesis. Indicate where the Induction Hypothesis is used in your proof.

(a) $a_3 = 3a_2 + 4a_1 = 3\cdot 4 + 4\cdot 1 = 4\cdot 4 = 16 = 2^4$

$a_4 = 3a_3 + 4a_2 = 3\cdot 16 + 4\cdot 4 = 4\cdot 16 = 64 = 2^6$

$a_5 = 3a_4 + 4a_3 = 3\cdot 64 + 4\cdot 16 = 4\cdot 64 = 2^8$

$a_1 = 2^0$

$a_2 = 2^2$

(b) $a_n = 2^{2n-2}$

(c) Let $P(n)$: "$a_n = 2^{2n-2}$" for $n \geq 1$.

Additional work space. Please do not detach.

BI: to prove $P(1)$: "$a_1 = 2^{2\cdot 1-2}$"

LHS: $a_1 = 1$

RHS: $2^{2\cdot 1-2} = 2^0 = 1$

As LHS = RHS, $P(1)$ is T.

IS: to prove $P(1) \wedge P(2) \wedge \ldots P(n) \Rightarrow P(n+1)$ for all $n \geq 1$.

Fix $n \geq 1$. Assume

$P(1) \wedge P(2) \wedge \ldots \wedge P(n)$: "$a_k = 2^{2k-2}$ for all $1 \leq k \leq n$" (IH)

Examine $P(n+1)$: "$a_{n+1} = 2^{2(n+1)-2}$"

Case 1: $n \geq 2$. Then

LHS: $a_{n+1} = 3a_n + 4a_{n-1}$

$\stackrel{IH}{=} 3\cdot 2^{2n-2} + 4\cdot 2^{2(n-1)-2} =$

$= 3\cdot 2^{2n-2} + 2^{2n-2} = 4\cdot 2^{2n-2} = 2^{2n}$

RHS: $2^{2(n+1)-2} = 2^{2n}$

As LHS = RHS, $P(n+1)$ follows in this case.

Case 2: $n = 1$. We prove $P(n+1)$ directly:

LHS: $a_2 = 4$

RHS: $2^{2\cdot 2-2} = 2^2 = 4$

As LHS = RHS, $P(2)$ is T.

Hence $P(n+1)$ follows in both cases.

By strong Induction, $P(n)$ is T for all $n \geq 1$.

[7pts] **(2)** Determine the coefficient of x^9 in the expansion of $(x^3 - \frac{1}{x^4})^{10}$.

Your final answer should include no unevaluated binomial coefficients or factorials.

By the Binomial theorem:

$$\left(x^3 - \frac{1}{x^4}\right)^{10} = \sum_{i=0}^{10} \binom{10}{i} (x^3)^i \left(-x^{-4}\right)^{10-i}$$

$$= \sum_{i=0}^{10} \binom{10}{i} (-1)^{10-i} x^{3i} x^{-40+4i}$$

$$= \sum_{i=0}^{10} \binom{10}{i} (-1)^{10-i} x^{-40+7i}$$

The coefficient of x^9 occurs for i s.t. $-40+7i = 9$, that is, $i = 7$.

This coefficient is then

$$\binom{10}{7}(-1)^{10-7} = \binom{10}{3}(-1)^3 = -\frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}$$

$$= -5 \cdot 3 \cdot 8 = -120$$

[7pts] **(2)** Determine the coefficient of x^{15} in the expansion of $(x^4 - \frac{1}{x^3})^9$.

Your final answer should include no unevaluated binomial coefficients or factorials.

By the Binomial theorem:

$$\left(x^4 - \frac{1}{x^3}\right)^9 = \sum_{i=0}^{9} \binom{9}{i} (x^4)^i (-x^{-3})^{9-i}$$

$$= \sum_{i=0}^{9} \binom{9}{i} (-1)^{9-i} x^{4i} x^{-27+3i}$$

$$= \sum_{i=0}^{9} \binom{9}{i} (-1)^{9-i} x^{-27+7i}$$

The coefficient of x^{15} occurs for i s.t. $-27+7i = 15$

that is, $i = 6$

This coefficient is then

$$\binom{9}{6}(-1)^{9-6} = \binom{9}{3}(-1)^3 = -\frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}$$

$$= -3\cdot 4\cdot 7 = -84$$

[7pts] **(3)** Let $A, B, C \subseteq \mathcal{U}$.

(a) Prove that if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cup B$ and $C \subseteq A \cap B$.

(b) Is it true that for all $A, B, C \subseteq \mathcal{U}$,

$$C \subseteq A \cup B \Longrightarrow C \subseteq A \text{ or } C \subseteq B?$$

If so, give a proof. Otherwise, give a counterexample, and briefly explain why this is a counterexample.

(c) Is it true that for all $A, B, C \subseteq \mathcal{U}$,

$$C \subseteq A \cap B \Longrightarrow C \subseteq A \text{ and } C \subseteq B?$$

If so, give a proof. Otherwise, give a counterexample, and briefly explain why this is a counterexample.

(a) Assume $C \subseteq A$ and $C \subseteq B$.

Take any $x \in C$.

$\Rightarrow x \in A$ since $C \subseteq A$

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in A \cup B$

So $C \subseteq A \cup B$

Take any $x \in C$

$\Rightarrow x \in A$ and $x \in B$ since $C \subseteq A$ and $C \subseteq B$

$\Rightarrow x \in A \cap B$

So $C \subseteq A \cap B$.

(b) False. Counterexample: $A = \{1,2\}$, $B = \{2,3\}$, $C = \{1,3\}$.

Then $A \cup B = \{1,2,3\}$ so $C \subseteq A \cup B$

but $C \nsubseteq A$ and $C \nsubseteq B$.

Additional work space. Please do not detach.

(c) True. Proof:

Assume $C \subseteq A \cap B$.

Take any $x \in C$.

$\Rightarrow x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$ $\quad (*)$

$\Rightarrow x \in A$

so $C \subseteq A$

Also: $(*) \Rightarrow x \in B$

so $C \subseteq B$.

[7pts] **(4)** Let $A = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define a relation $\sim$ on A as follows:

$$(a, b) \sim (c, d) \iff ad = bc.$$

(a) Prove that $\sim$ is an equivalence relation on A.

(b) Determine the equivalence class $[(4, 7)]_\sim$. *Use the set-builder notation for this set, describing its elements as explicitly as possible.*

(c) Give two concrete elements of the equivalence class $[(4, 7)]_\sim$.

(a) $\sim$ is reflexive: for all $(a,b) \in A$,

$ab = ba \Rightarrow (a,b) \sim (a,b)$

$\sim$ is symmetric: for all $(a,b), (c,d) \in A$,

$(a,b) \sim (c,d) \Rightarrow ad = bc \Rightarrow cb = da$

$\Rightarrow (c,d) \sim (a,b)$

$\sim$ is transitive: for all $(a,b), (c,d), (e,f) \in A$,

$(a,b) \sim (c,d) \wedge (c,d) \sim (e,f)$

$\Rightarrow ad = bc \wedge cf = de$

$\Rightarrow af = ad\frac{f}{d} = bc\frac{f}{d} = b\frac{cf}{d} = be$

$\Rightarrow (a,b) \sim (e,f)$

(b) $[(4,7)]_\sim = \{(a,b) \in A : (a,b) \sim (4,7)\}$

$= \{(a,b) \in A : 7a = 4b\}$

(c) For example, $(4,7), (8,14), (-4,-7) \in [(4,7)]_\sim$

$[(5,7)]_\sim = \{(a,b) \in A : 7a = 5b\}$, e.g. $(5,7), (10,14), (-5,-7) \in [(5,7)]_\sim$

[7pts] **(5)** (a) Determine the following:

(i) $[6334] \oplus [2402]$ in $\mathbb{Z}_3$

(ii) $[28289] \odot [999328521]$ in $\mathbb{Z}_2$

For each of the subquestions above, express your final answer using the canonical representative (that is, as one of $[0], [1], \ldots, [n-1]$, *where* $n \in \mathbb{N}$ *is the module).*

(b) Does there exist an element of $\mathbb{Z}_6$ that admits a multiplicative inverse? If so, find all such elements, and for each, determine its multiplicative inverse. If not, explain why. *Justify all your calculations.*

(a) $[6334] \oplus [2402] = [1] \oplus [2] = [1+2] = [0]$ in $\mathbb{Z}_3$

since $6334 = 2111 \cdot 3 + 1$

and $2402 = 800 \cdot 3 + 2$

$[28289] \odot [999328521] = [1] \odot [1] = [1]$ in $\mathbb{Z}_2$

since $[28289] = [1]$ as 28289 is odd

and $[999328521] = [1]$ as 999328521 is odd.

(b) We are looking for $[a], [b] \in \mathbb{Z}_6$ s.t. $[a] \odot [b] = [1]$.

Now $[a] \odot [b] = [1] \implies [ab] = [1]$

$\implies 6 \mid (ab-1)$

$\implies 2 \mid (ab-1)$ and $3 \mid (ab-1)$

$\implies 2 \nmid a$ and $3 \nmid a$

We are left with $a \in \{1, 5\}$.

We aare $1 \cdot 1 - 1 = 0$ so $6 \mid (1 \cdot 1 - 1)$

and $5 \cdot 5 - 1 = 24$ so $6 \mid (5 \cdot 5 - 1)$

Additional work space. Please do not detach.

So the elements of $\mathbb{Z}_6$ with a mult. inverse are:

[1] with mult. inverse [1]

[5] " " [5]

Alternatively, the multiplication table for $\mathbb{Z}_6$ is:

	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]		[1]	[2]	[3]	[4]	[5]
[2]			[4]	[0]	[2]	[4]
[3]				[3]	[0]	[3]
[4]					[4]	[2]
[5]						[1]