Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Mathematical Reasoning and Proofs MAT1362

First Midterm Exam (α)

8 February 2024

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Instructions. *You must sign below to confirm that you have read, understand, and will follow them.*

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 5 questions on 8 pages. Page 8 contains the Table of Logical Equivalences. *Please do not detach it.*
- Be sure to read carefully and follow the instructions for the individual problems. To receive full marks, your solution/proof must be correct, complete, and show all relevant details.
- For rough work or additional work space, you may use the back pages. Do not use scrap paper of your own.
- Use correct mathematical notation and terminology, as defined in class.
- You may ask for clarification.
- Unauthorized electronic devices (such as cellular phones) are not permitted during this exam. Such devices must be *turned off completely* and stored out of students' reach. Students found in possession of such a device during the exam will be asked to leave immediately and academic fraud allegations may be filed.

LAST NAME:______________________________

First name:______________________________

Signature:______________________________

Write your student number on the next page.

Circle your DGD (this is where you will pick up your marked exam):

D01	**D02**
11:30	13:00
MRT 015	MRT 219

Student number:__________________

Question	1	2	3	4	5	Total
Max	7	7	7	7	7	35
Marks						

[7pts] **(1)** Let P and Q be two logical propositions.

(a) Use a **truth table** to prove the Absorption Law: $P \vee (P \wedge Q) \equiv P$

(b) Use the **Table of Logical Equivalences** on p. 8 to prove the following equivalence:

$$\Big((P \vee \neg Q) \implies (Q \wedge \neg Q)\Big) \equiv (\neg P \wedge Q)$$

Use exactly one equivalence per step, and name it, too.

[7pts] **(2)** Use **Mathematical Induction** to prove the following:

$$(\forall n \in \mathbb{N}) \left(\sum_{j=1}^{n} (6j-2) = n(3n+1) \right).$$

Clearly state the proposition to be proved, Basis of Induction, Induction Step, and Induction Hypothesis. Indicate clearly where the Induction Hypothesis is used in your proof.

[7pts] **(3)** (a) Consider the following proposition:

"If I am happy and I sing, then my brother gets annoyed."

In words, write the **converse** and the **contrapositive** of this proposition.

Converse:

Contrapositive:

(b) In words, write the **negation** of the proposition

"I am happy only if I sing."

Negation:

(c) Symbolically, write the **negation** of the proposition

$$(\forall x \in \mathbb{Z})(\exists y, z \in \mathbb{Z} \text{ s.t.})\ (x \geq 2y + 3z)$$

Simplify your negation as much as possible, in particular, so that no negation signs remain.

Negation:

[7pts] **(4)** Prove the following propositions using only the **Integer Axioms**.

(a) **Proposition**: Let $a \in \mathbb{Z}$. If $ba = b$ for all $b \in \mathbb{Z}$, then $a = 1$.

(b) **Proposition**: Let $a \in \mathbb{Z}$. Then $-(-a) = a$.

Careful: your proof must consist of a series of steps that logically follow from previous steps. Each step must be justified by a single Integer Axiom (from the list presented in class), or by a property of the relation "=". Name each axiom being used (e.g. "associativity of multiplication").

(5) Use an appropriate type of proof to prove the following.

Proposition: Let $a, b \in \mathbb{Z}$. Then $-a < -b \Longleftrightarrow a > b$.

You may use only the axioms of $\mathbb{N}$ (including axioms of $\mathbb{Z}$), the definition of the relation "$<$", the definition of subtraction, and any propositions already proved on this exam. Careful: do not use any propositions proved in class!

Table of Logical Equivalences

	Equivalence	Name
(1)	$P \Rightarrow Q \equiv \neg P \vee Q$	Implication Law
(2)	$P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	Biconditional Laws
(3)	$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$	
(4)	$P \vee \neg P \equiv \mathbf{T}$	Negation Laws
(5)	$P \wedge \neg P \equiv \mathbf{F}$	
(6)	$P \vee \mathbf{F} \equiv P$	Identity Laws
(7)	$P \wedge \mathbf{T} \equiv P$	
(8)	$P \vee \mathbf{T} \equiv \mathbf{T}$	Domination Laws
(9)	$P \wedge \mathbf{F} \equiv \mathbf{F}$	
(10)	$P \vee P \equiv P$	Idempotent Laws
(11)	$P \wedge P \equiv P$	
(12)	$\neg\neg P \equiv P$	Double negation
(13)	$P \vee Q \equiv Q \vee P$	Commutative Laws
(14)	$P \wedge Q \equiv Q \wedge P$	
(15)	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	Associative Laws
(16)	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
(17)	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	Distributive Laws
(18)	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	
(19)	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	De Morgan's Laws
(20)	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	