Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Mathematical Reasoning and Proofs MAT1362

First Midterm Exam (α)

8 February 2024

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Instructions. *You must sign below to confirm that you have read, understand, and will follow them.*

- This is an 80-minute *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 5 questions on 8 pages. Page 8 contains the Table of Logical Equivalences. *Please do not detach it.*
- Be sure to read carefully and follow the instructions for the individual problems. To receive full marks, your solution/proof must be correct, complete, and show all relevant details.
- For rough work or additional work space, you may use the back pages. Do not use scrap paper of your own.
- Use correct mathematical notation and terminology, as defined in class.
- You may ask for clarification.
- Unauthorized electronic devices (such as cellular phones) are not permitted during this exam. Such devices must be *turned off completely* and stored out of students' reach. Students found in possession of such a device during the exam will be asked to leave immediately and academic fraud allegations may be filed.

LAST NAME: SOLUTIONS

First name: ____________

Signature: ____________

Write your student number on the next page.

Circle your DGD (this is where you will pick up your marked exam):

D01	D02
11:30	13:00
MRT 015	MRT 219

Student number:________________

Question	1	2	3	4	5	Total
Max	7	7	7	7	7	35
Marks						

[7pts] **(1)** Let P and Q be two logical propositions.

(a) Use a **truth table** to prove the Absorption Law: $P \vee (P \wedge Q) \equiv P$

P	Q	$P \wedge Q$	$P \vee (P \wedge Q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

$\equiv$

(b) Use the **Table of Logical Equivalences** on p. 8 to prove the following equivalence:

$$\Big((P \vee \neg Q) \Longrightarrow (Q \wedge \neg Q)\Big) \equiv (\neg P \wedge Q)$$

Use exactly one equivalence per step, and name it, too.

$(P \vee \neg Q) \Rightarrow (Q \wedge \neg Q)$

$\equiv (P \vee \neg Q) \Rightarrow F$ Negation Law

$\equiv \neg (P \vee \neg Q) \vee F$ Implication Law

$\equiv \neg (P \vee \neg Q)$ Identity Law

$\equiv \neg P \wedge \neg (\neg Q)$ DeMorgan's Law

$\equiv \neg P \wedge Q$ Double Negation

[7pts] **(2)** Use **Mathematical Induction** to prove the following:

$$(\forall n \in \mathbb{N}) \left(\sum_{j=1}^{n} (6j-2) = n(3n+1) \right).$$

Clearly state the proposition to be proved, Basis of Induction, Induction Step, and Induction Hypothesis. Indicate clearly where the Induction Hypothesis is used in your proof.

Let $P(n)$: " $\sum_{j=1}^{n} (6j-2) = n(3n+1)$ "

We must prove $P(n)$ for all $n \in \mathbb{N}$.

BI: to prove $P(1)$: " $\sum_{j=1}^{1} (6j-2) = 1\cdot(3\cdot 1+1)$ "

LHS: $\sum_{j=1}^{1} (6j-2) = 6\cdot 1 - 2 = 4$ RHS: $1\cdot(3\cdot 1+1) = 4$

So LHS = RHS, i.e. $P(1)$ is T.

IS: To prove $P(n) \Rightarrow P(n+1)$ for all $n \geq 1$.

Fix $n \geq 1$. Assume $P(n)$: " $\sum_{j=1}^{n} (6j-2) = n(3n+1)$ " (IH)

show $P(n+1)$ follows:

$$\text{LHS} = \sum_{j=1}^{n+1} (6j-2) = \sum_{j=1}^{n} (6j-2) + 6(n+1) - 2$$

$$\overset{\text{IH}}{=} n(3n+1) + 6(n+1) - 2$$

$$= 3n^2 + n + 6n + 4 = 3n^2 + 7n + 4$$

$$\text{RHS} = (n+1)(3(n+1)+1) = (n+1)(3n+4) = 3n^2 + 7n + 4 = \text{LHS}$$

Hence $P(n+1)$ follows.

Conclusion: since $P(1)$ is T, and $P(n) \Rightarrow P(n+1)$ is T for all $n \in \mathbb{N}$, by PMI, $P(n)$ is T for all $n \in \mathbb{N}$.

[7pts] **(3)** (a) Consider the following proposition:

"If I am happy and I sing, then my brother gets annoyed."

In words, write the **converse** and the **contrapositive** of this proposition.

Converse: If my brother gets annoyed, then I am happy and I sing.

Contrapositive: If my brother does not get annoyed, then I am not happy or I do not sing.

(b) In words, write the **negation** of the proposition

"$\underbrace{\text{I am happy}}_{P}$ only if $\underbrace{\text{I sing}}_{Q}$." i.e. $P \Rightarrow Q$

Negation: $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$

Negation:

I am happy and I do not sing.

(c) Symbolically, write the **negation** of the proposition

$$(\forall x \in \mathbb{Z})(\exists y, z \in \mathbb{Z} \text{ s.t.})\ (x \geq 2y + 3z)$$

Negation:

$$\neg (\forall x \in \mathbb{Z})(\exists y, z \in \mathbb{Z} \text{ s.t.})(x \geq 2y + 3z)$$

$$\equiv (\exists x \in \mathbb{Z} \text{ s.t.}) \neg (\exists y, z \in \mathbb{Z} \text{ s.t.})(x \geq 2y + 3z)$$

$$\equiv (\exists x \in \mathbb{Z} \text{ s.t.})(\forall y, z \in \mathbb{Z}) \neg (x \geq 2y + 3z)$$

$$\equiv (\exists x \in \mathbb{Z} \text{ s.t.})(\forall y, z \in \mathbb{Z})(x < 2y + 3z)$$

[7pts] **(4)** Prove the following propositions using only the **Integer Axioms**.

(a) **Proposition**: Let $a \in \mathbb{Z}$. If $ba = b$ for all $b \in \mathbb{Z}$, then $a = 1$.

(b) **Proposition**: Let $a \in \mathbb{Z}$. Then $-(-a) = a$.

Careful: your proof must consist of a series of steps that logically follow from previous steps. Each step must be justified by a single Integer Axiom (from the list presented in class), or by a property of the relation "=". Name each axiom being used (e.g. "commutativity of addition").

(a) Let $a \in \mathbb{Z}$. Assume $ba = b$ for all $b \in \mathbb{Z}$.
By the multiplicative identity axiom, $1 \in \mathbb{Z}$.
Hence $1 \cdot a = 1$
$\Rightarrow$ $a \cdot 1 = 1$ (commutativity of multiplication)
$\Rightarrow$ $a = 1$ (multiplicative identity) $\square$

(b) Let $a \in \mathbb{Z}$. Then
$a + (-a) = 0$ (additive inverse axiom)
$\Rightarrow$ $(-a) + a = 0$ (commutativity of addition)
$\Rightarrow$ $(-a) + a = (-a) + (-(-a))$ (additive inverse axiom)
$\Rightarrow$ $a + ((-a) + a) = a + ((-a) + (-(-a)))$ (replacement prop.)
$\Rightarrow$ $(a + (-a)) + a = (a + (-a)) + (-(-a))$ (associativity of addition)
$\Rightarrow$ $0 + a = 0 + (-(-a))$ (additive inverse axiom)
$\Rightarrow$ $a + 0 = (-(-a)) + 0$ (commutativity of addition)
$\Rightarrow$ $a = -(-a)$ (additive inverse axiom)

(5) Use an appropriate type of proof to prove the following.

Proposition: Let $a, b \in \mathbb{Z}$. Then $\underbrace{-a < -b}_{P} \iff \underbrace{a > b}_{Q}$.

You may use only the axioms of $\mathbb{N}$, *the definition of the relation "$<$", the definition of subtraction, and any propositions already proved on this exam.*

Careful: do not use any propositions proved in class!

We need to prove $P \Leftrightarrow Q$. Using a proof of equivalence, we need to show $P \Rightarrow Q$ and $Q \Rightarrow P$.

<u>To show $P \Rightarrow Q$:</u>

Assume $-a < -b$. Then $(-b)-(-a) \in \mathbb{N}$ (defn. of "$<$")

Now $(-b)-(-a) = (-b)+(-(-a))$ (defn. of subtraction)

$= (-b)+a$ (by A4b)

$= a+(-b)$ (commutativity of add.)

$= a-b$ (defn. of subtraction)

Hence $a-b \in \mathbb{N}$. By defn. of $<$, we conclude $a>b$.

<u>To show $Q \Rightarrow P$:</u>

Assume $a>b$. Then $a-b \in \mathbb{N}$ (defn. of "$>$").

Now $a-b = a+(-b)$ (defn. of subtraction)

$= (-b)+a$ (commutativity of addition)

$= (-b)+(-(-a))$ (by A4b)

$= (-b)-(-a)$ (defn. of subtraction)

Hence $(-b)-(-a) \in \mathbb{N}$.

By defn. of "$<$", we conclude $-a<-b$.

Table of Logical Equivalences

	Equivalence	Name
(1)	$P \Rightarrow Q \equiv \neg P \vee Q$	Implication Law
(2)	$P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	Biconditional Laws
(3)	$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$	
(4)	$P \vee \neg P \equiv \mathbf{T}$	Negation Laws
(5)	$P \wedge \neg P \equiv \mathbf{F}$	
(6)	$P \vee \mathbf{F} \equiv P$	Identity Laws
(7)	$P \wedge \mathbf{T} \equiv P$	
(8)	$P \vee \mathbf{T} \equiv \mathbf{T}$	Domination Laws
(9)	$P \wedge \mathbf{F} \equiv \mathbf{F}$	
(10)	$P \vee P \equiv P$	Idempotent Laws
(11)	$P \wedge P \equiv P$	
(12)	$\neg\neg P \equiv P$	Double negation
(13)	$P \vee Q \equiv Q \vee P$	Commutative Laws
(14)	$P \wedge Q \equiv Q \wedge P$	
(15)	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	Associative Laws
(16)	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
(17)	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	Distributive Laws
(18)	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	
(19)	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	De Morgan's Laws
(20)	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	