Université d'Ottawa · University of Ottawa
Faculté des sciences Mathématiques et de statistique
Faculty of Science Mathematics and Statistics

Mathematical Reasoning and Proofs MAT1362D

Final Exam

25 April 2024

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You must **sign below** *to confirm that you have read, understand, and will follow* **these instructions:**

- This is an 3-hour *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 11 questions (each worth 10 marks) on 21 pages. Page 21 contains the Table of Logical Equivalences. *Please do not detach it.*
- Be sure to read carefully and follow the instructions for the individual problems. To receive full marks, your solution/proof must be correct, complete, and show all relevant details.
- For rough work or additional work space, you may use the back pages. You may also ask the proctors for scrap paper, but do not use scrap paper of your own.
- Use correct mathematical notation and terminology, as defined in class.
- You may ask for clarification.
- Unauthorized electronic devices (such as cellular phones and smart watches) are not permitted during this exam. Such devices must be *turned off completely* and stored out of students' reach. Do not keep them in your possession, such as in your pockets. If you are caught with such a device, academic fraud allegations may be filed which may result in your obtaining a 0 (zero) for the exam.

LAST NAME:____________________________

First name:____________________________

Student number:____________________________

Signature:____________________________

Question	1	2	3	4	5	6	7	8	9	10	11	Total
Max	10	10	10	10	10	10	10	10	10	10	10	110
Marks												

(Q1) Let P, Q, R be propositional variables.

[5pts] (a) Use a **truth table** to determine whether or not the compound propositions

$$(P \wedge Q) \Rightarrow R \quad \text{and} \quad (P \Rightarrow R) \wedge (Q \Rightarrow R)$$

are logically equivalent. *Clearly state your conclusion, and justify it referring to the truth table.*

[5pts] (b) Use the **Table of Logical Equivalences** on p. 21 to prove the following equivalence:

$$\Big((P \Rightarrow R) \wedge (Q \Rightarrow R)\Big) \equiv \Big((P \vee Q) \Rightarrow R)\Big)$$

Use exactly one equivalence per step, and name it, too.

(Q2) (a) Let $A, B \subseteq \mathbb{R}$, and let P be the following proposition:

[4.5pts]

P : "If A is a subset of B and B is bounded above, then A is empty or $\sup(A)$ exists."

State (i) the contrapositive, (ii) the converse, and (iii) the negation of P. *Use words, that is, write these propositions in the same style as P is written above.*

(i) the contrapositive of P:

(ii) the converse of P:

(iii) the negation of P:

(b) For each of the following propositions, determine whether it is true or false, and then state the negation.

[5.5pts]

You need not prove/disprove the proposition. For the negation, use quantifiers, but simplify the quantified statement so that no symbols $\neg$ and $\nexists$ remain.

(i) $(\exists N \in \mathbb{N} \text{ s.t.})(\forall n \in \mathbb{N})(n \leq N)$ *Circle:* T F

Negation:

(ii) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R} \text{ s.t.})(x + y = 0)$ *Circle:* T F

Negation:

(iii) $(\forall x, y \in \mathbb{R})(\exists z \in \mathbb{R} \text{ s.t.})(x < y \Rightarrow x < z < y)$ *Circle:* T F

Negation:

(Q3) Let $a \in \mathbb{Z}$. Using only the axioms of $\mathbb{Z}$, prove the following two propositions.

Use one axiom per step, and name it, too.

[5pts] (a) $a \cdot 0 = 0 \cdot a = 0$.

[5pts] (b) If $b + a = b$ for some $b \in \mathbb{Z}$, then $a = 0$.

(Q4) Let $(f_j)_{j=1}^{\infty}$ be a sequence in $\mathbb{Z}$ defined recursively as follows:

$$f_1 = 0, \quad f_2 = 1, \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \quad \text{for all } n \geq 3.$$

[3pts] (a) Determine f_3, f_4, and f_5.

[7pts] (b) Use Strong Induction to prove that for all integers $n \geq 2$,

$$f_{n+3} = 3f_n + 2f_{n-1}.$$

Important: clearly state the proposition to be proved, the Basis of Induction, the Induction Step, and the Induction Hypothesis. Indicate where the Induction Hypothesis is used in your proof.

Additional work space. Please do not detach.

(Q5) Let $A, B, C \subseteq \mathcal{U}$.

[3pts] (a) Give the precise definition of sets $A \cap B$ and $A \cup B$, using the set-builder notation.

[7pts] (b) For each of the following statements, determine whether it is true or false (for all A, B, C). If you claim that it is true, give a rigorous proof using the definition of set operations; otherwise, give a concrete counterexample and demonstrate that this is a counterexample. *(Do not use set identities.)*

(i) If $A \cup C = B \cup C$, then $A = B$.

(ii) If $A \cap C = B \cap C$, then $A = B$.

(iii) If $A \cup C = B \cup C$ and $A \cap C = B \cap C$, then $A = B$.

Additional work space. Please do not detach.

(Q6) A relation R on the set $\mathbb{Z}$ is defined as follows:

$$xRy \iff 4|(x^2 - y^2).$$

[5pts] (a) Prove that R is an equivalence relation.

(b) Describe [0] and [1], that is, the equivalence classes of 0 and 1, respectively.

[2pts] *Use the set-builder notation, and be as explicit as possible.*

[3pts] (c) Show that each $n \in \mathbb{Z}$ is either an element of [0] or an element of [1].

Additional work space. Please do not detach.

(Q7) (a) Give the full statement for each of the five axioms that we used to define the set of real numbers, $\mathbb{R}$, as prompted below:

[5pts]

- Axiom 7.1:

  (i) Commutativity of Addition:

  (ii) Associativity of Addition:

  (iii) Distributivity:

  (iv) Commutativity of Multiplication:

  (v) Associativity of Multiplication:

- Axiom 7.2: Additive Identity Axiom:

- Axiom 7.3: Multiplicative Identity Axiom:

- Axiom 7.4: Additive Inverse Axiom:

- Axiom 7.5: Multiplicative Inverse Axiom:

[5pts] (b) Using only the five axioms from (a) and the replacement property, prove the *multiplicative cancellation property* for real numbers:

For all $x, y, z \in \mathbb{R}$ such that $x \neq 0$, if $xy = xz$, then $y = z$.

Use one axiom per step, and name it, too.

(Q8) Let $f : A \longrightarrow B$ be a function.

[2pts] (a) Give a precise definition that explains what is meant by "f is *injective*".

[2pts] (b) Give a precise definition that explains what is meant by "f is *surjective*".

[6pts] (c) Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $f(n) = 3n - 2$.

(i) Prove that f is injective.

(ii) Prove that f is not surjective.

(iii) Find a left inverse g of f.
Be sure to verify that your g is indeed a left inverse of f.

Additional work space. Please do not detach.

(Q9) Let A be a subset of $\mathbb{R}$ defined as

$$A = \left\{ 5 - \frac{2}{n} : n \in \mathbb{N} \right\}.$$

Fully justify all your answers below. In this question, you may use the arithmetic of $\mathbb{R}$ *without referring to axioms or propositions.*

[4pts] (a) Find the minimum of A, or else prove that it does not exist.

[2pts] (b) Find the infimum of A, or else prove that it does not exist.

[4pts] (c) Find the supremum of A, or else prove that it does not exist.

Additional work space. Please do not detach.

(Q10) (a) Let $(x_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$, and $L \in \mathbb{R}$. Give a precise definition that explains
[3pts] what is meant by "the sequence $(x_k)_{k=1}^{\infty}$ *converges* to L".

[7pts] (b) Determine $\lim_{k\to\infty} \frac{2k-1}{k+3}$.
You must prove that your answer is correct using the definition of a limit of a sequence from (a), and using no other results on limits.

(Q11) Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ defined recursively as follows:

$$x_1 = 1, \quad \text{and} \quad x_n = \frac{1}{3}(x_{n-1} + 6) \quad \text{for all } n \geq 2.$$

[6pts] (a) Using induction, prove that this sequence is bounded above by 3 and bounded below by 0; that is, prove that for all $n \in \mathbb{N}$,

$$0 \leq x_n \leq 3.$$

[4pts] (b) Prove that this sequence is increasing.

[3pts] (c) **(Bonus)** Is this sequence convergent? Justify your answer.

Additional work space. Please do not detach.

Table of Logical Equivalences

	Equivalence	Name
(1)	$P \Rightarrow Q \equiv \neg P \vee Q$	Implication Law
(2)	$P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	Biconditional Laws
(3)	$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$	
(4)	$P \vee \neg P \equiv \mathbf{T}$	Negation Laws
(5)	$P \wedge \neg P \equiv \mathbf{F}$	
(6)	$P \vee \mathbf{F} \equiv P$	Identity Laws
(7)	$P \wedge \mathbf{T} \equiv P$	
(8)	$P \vee \mathbf{T} \equiv \mathbf{T}$	Domination Laws
(9)	$P \wedge \mathbf{F} \equiv \mathbf{F}$	
(10)	$P \vee P \equiv P$	Idempotent Laws
(11)	$P \wedge P \equiv P$	
(12)	$\neg\neg P \equiv P$	Double negation
(13)	$P \vee Q \equiv Q \vee P$	Commutative Laws
(14)	$P \wedge Q \equiv Q \wedge P$	
(15)	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	Associative Laws
(16)	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
(17)	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	Distributive Laws
(18)	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	
(19)	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	De Morgan's Laws
(20)	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	