Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Mathematical Reasoning and Proofs MAT1362

Final Exam

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Instructions. *You must sign below to confirm that you have read, understand, and will follow them.*

- This is an 3-hour *closed-book* exam; no notes are allowed. Calculators are *not* permitted.
- The exam consists of 11 questions (each worth 10 marks) on 20 pages. Page 20 contains the Table of Logical Equivalences. *Please do not detach it.*
- Be sure to read carefully and follow the instructions for the individual problems. To receive full marks, your solution/proof must be correct, complete, and show all relevant details.
- For rough work or additional work space, you may use the back pages. Do not use scrap paper of your own.
- Use correct mathematical notation and terminology, as defined in class.
- You may ask for clarification.
- Unauthorized electronic devices (such as cellular phones and smart watches) are not permitted during this exam. Such devices must be *turned off completely* and stored out of students' reach. Students found in possession of such a device during the exam will be asked to leave immediately and academic fraud allegations may be filed.

LAST NAME:____________________

First name:____________________

Student number:____________________

Signature:____________________

Question	1	2	3	4	5	6	7	8	9	10	11	Total
Max	10	10	10	10	10	10	10	10	10	10	10	110
Marks												

[10pts] **(Q1)** Let P, Q, R be propositional variables.

(a) Use a **truth table** to determine whether or not the compound propositions

$$(P \wedge Q) \Rightarrow R \quad \text{and} \quad (P \Rightarrow R) \wedge (Q \Rightarrow R)$$

are logically equivalent. *Clearly state your conclusion, and justify it referring to the truth table.*

P	Q	R	$P \wedge Q$	$P \wedge Q \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

$\not\equiv$

Conclusion: $P \wedge Q \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are <u>not</u> equivalent.

(b) Use the **Table of Logical Equivalences** on p. ?? to prove the following equivalence:

$$\Big((P \Rightarrow R) \wedge (Q \Rightarrow R)\Big) \equiv \Big((P \vee Q) \Rightarrow R\Big)$$

Use exactly one equivalence per step, and name it, too.

$$\begin{aligned}
(P \Rightarrow R) \wedge (Q \Rightarrow R) &\equiv (\neg P \vee R) \wedge (\neg Q \vee R) && \text{(Implication)}\\
&\equiv (R \vee \neg P) \wedge (R \vee \neg Q) && \text{(Commutative)}\\
&\equiv R \vee (\neg P \wedge \neg Q) && \text{(Distributive)}\\
&\equiv (\neg P \wedge \neg Q) \vee R && \text{(Commutative)}\\
&\equiv \neg (P \vee Q) \vee R && \text{(De Morgan's)}\\
&\equiv (P \vee Q) \Rightarrow R && \text{(Implication)}
\end{aligned}$$

[10pts] **(Q2)** (a) Let $A, B \subseteq \mathbb{R}$, and let P be the following proposition:

P : "If A is a subset of B and B is bounded above, then A is empty or $\sup(A)$ exists."

State (i) the contrapositive, (ii) the converse, and (iii) the negation of P. *Use words, that is, write these propositions in the same style as P is written above.*

(i) the contrapositive of P:

"If A is nonempty and sup(A) does not exist, then A is not a subset of B or B is not bounded above."

(ii) the converse of P:

"If A is empty or sup(A) exists, then A is a subset of B and B is bounded above."

(iii) the negation of P:

"A is a subset of B and B is bounded above, and A is nonempty, and sup(B) does not exist."

(b) For each of the following propositions, determine whether it is true or false, and then state the negation.

You need not prove/disprove the proposition. For the negation, use quantifiers, but simplify the quantified statement so that no symbols $\neg$ and $\nexists$ remain.

(i) $(\exists N \in \mathbb{N} \text{ s.t.})(\forall n \in \mathbb{N})(n \leq N)$ *Circle:* T F

Negation: $(\forall N \in \mathbb{N})(\exists n \in \mathbb{N} \text{ s.t.})(n > N)$

(ii) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R} \text{ s.t.})(x + y = 0)$ *Circle:* T F

Negation: $(\exists x \in \mathbb{R} \text{ s.t.})(\forall y \in \mathbb{R})(x + y \neq 0)$

(iii) $(\forall x, y \in \mathbb{R})(\exists z \in \mathbb{R} \text{ s.t.})(x < y \Rightarrow x < z < y)$ *Circle:* T F

Negation: $(\exists x, y \in \mathbb{R} \text{ s.t.})(\forall z \in \mathbb{R})(x < y \wedge (z \leq x \vee z \geq y))$

(iv) $(\forall A \subseteq \mathbb{R})(\exists M \in \mathbb{R} \text{ s.t.})(\forall a \in A)(a \leq M)$ *Circle:* T F

Negation: $(\exists A \subseteq \mathbb{R} \text{ s.t.})(\forall M \in \mathbb{R})(\exists a \in A)(a > M)$

(Q3) [10pts] Let $a \in \mathbb{Z}$. Using only the axioms of $\mathbb{Z}$, prove the following two propositions. Use one axiom per step, and name it, too.

(a) $a \cdot 0 = 0 \cdot a = 0$.

(b) If $b + a = b$ for some $b \in \mathbb{Z}$, then $a = 0$.

(a) $0 = 0 + 0$ (additive identity)

$\Rightarrow a \cdot 0 = a \cdot (0 + 0)$ (replacement property)

$\Rightarrow a \cdot 0 = a \cdot 0 + a \cdot 0$ (distributivity)

$\Rightarrow a \cdot 0 + (-(a \cdot 0)) = (a \cdot 0 + a \cdot 0) + (-(a \cdot 0))$ (replacement)

$\Rightarrow a \cdot 0 + (-(a \cdot 0)) = a \cdot 0 + (a \cdot 0 + (-(a \cdot 0)))$ (associativity +)

$\Rightarrow 0 = a \cdot 0 + 0$ (add. inverse)

$\Rightarrow 0 = a \cdot 0$ (add. identity)

$\Rightarrow 0 = 0 \cdot a$ (commutativity $\cdot$)

(b) Assume $b + a = b$ for some $b \in \mathbb{Z}$.

$\Rightarrow (-b) + (b + a) = (-b) + b$ (replacement)

$\Rightarrow ((-b) + b) + a = (-b) + b$ (associativity +)

$\Rightarrow (b + (-b)) + a = b + (-b)$ (commutativity +)

$\Rightarrow 0 + a = 0$ (add. inverse)

$\Rightarrow a + 0 = 0$ (commutativity +)

$\Rightarrow a = 0$ (add. identity)

[10pts] **(Q4)** Let $(f_j)_{j=1}^{\infty}$ be a sequence in $\mathbb{Z}$ defined recursively as follows:

$$f_1 = 0, \quad f_2 = 1, \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \quad \text{for all } n \geq 3. \quad (*)$$

(a) Determine f_3, f_4, and f_5.

$f_3 = f_2 + f_1 = 1 \qquad f_4 = f_3 + f_2 = 2 \qquad f_5 = f_4 + f_3 = 3$

(b) Use Strong Induction to prove that for all integers $n \geq 2$,

$$f_{n+3} = 3f_n + 2f_{n-1}.$$

Clearly state the proposition to be proved, the Basis of Induction, the Induction Step, and the Induction Hypothesis. Indicate where the Induction Hypothesis is used in your proof.

Let $P(n)$: "$f_{n+3} = 3f_n + 2f_{n-1}$". We need to prove $P(n)$ for all $n \geq 2$.

BI: To prove $P(2)$: "$f_5 = 3f_2 + 2f_1$"

LHS: $f_5 = 3$

RHS: $3f_2 + 2f_1 = 3 \cdot 1 + 2 \cdot 0 = 3$

So LHS = RHS, and $P(2)$ is T.

IS: To prove $P(2) \wedge P(3) \wedge \dots \wedge P(n) \Rightarrow P(n+1)$ for all $n \geq 2$.

Fix any $n \geq 2$, and assume $P(2) \wedge \dots P(n)$ is T,

that is, "$f_{k+3} = 3f_k + 2f_{k-1}$ for all $2 \leq k \leq n$" (IH).

Examine $P(n+1)$: "$f_{n+4} = 3f_{n+1} + 2f_n$"

Case 1: $n \geq 3$

LHS: $f_{n+4} = f_{n+3} + f_{n+2}$ by $(*)$

$= (3f_n + 2f_{n-1}) + (3f_{n-1} + 2f_{n-2})$ by IH $(n \geq 3)$

$= 3(f_n + f_{n-1}) + 2(f_{n-1} + f_{n-2})$

$= 3f_{n+1} + 2f_n$ by (*)

$= RHS$

So $P(n+1)$ follows.

<u>Case 2</u>: $n=2$. It suffices to show $P(3)$: " $f_6 = 3f_3 + 2f_2$ ".

LHS : $f_6 = f_5 + f_4$ by (*)

$= 3 + 2 = 5$

RHS : $3f_3 + 2f_2 = 3 \cdot 1 + 2 \cdot 1 = 5$

So LHS = RHS and $P(3)$ is T.

Thus $P(n+1)$ follows in both cases.

Conclusion: since $P(2)$ is T, and $P(2) \wedge \ldots \wedge P(n) \Rightarrow P(n+1)$ is T for all $n \geq 2$, $P(n)$ is T for all $n \geq 2$.

[10pts] **(Q5)** Let $A, B, C \subseteq \mathcal{U}$.

(a) Give the precise definition of sets $A \cap B$ and $A \cup B$, using the set-builder notation.

$A \cap B = \{x \in \mathcal{U} : x \in A \text{ and } x \in B\}$

$A \cup B = \{x \in \mathcal{U} : x \in A \text{ or } x \in B\}$

(b) For each of the following statements, determine whether it is true or false (for all A, B, C). If you claim that it is true, give a rigorous proof using the definition of set operations; otherwise, give a concrete counterexample and demonstrate that this is a counterexample. *(Do not use set identities.)*

(i) If $A \cup C = B \cup C$, then $A = B$.

(ii) If $A \cap C = B \cap C$, then $A = B$.

(iii) If $A \cup C = B \cup C$ and $A \cap C = B \cap C$, then $A = B$.

(i) False. Counter example:

$A = \{1, 2\}$

$B = \{1, 2, 3\}$

$C = \{2, 3\}$

Then $A \cup C = \{1, 2, 3\} = B \cup C$ but $A \neq B$.

(ii) False. Counterexample:

$A = \{1, 2\}$

$B = \{1, 3\}$

$C = \{1\}$

Then $A \cap C = \{1\} = B \cap C$ but $A \neq B$

(iii) True. Proof:

Assume $A \cup C = B \cup C$ and $A \cap C = B \cap C$.

Take any $a \in A$

$\Rightarrow \quad a \in A \cup C$

$\Rightarrow \quad a \in B \cup C$

$\Rightarrow \quad a \in B \ \vee \ a \in C$

$\Rightarrow \quad a \in B \ \vee \ a \in A \cap C$ since $a \in A$

$\Rightarrow \quad a \in B \ \vee \ a \in B \cap C$ since $A \cap C = B \cap C$

$\Rightarrow \quad a \in B$

Hence $A \subseteq B$.

By symmetry, $B \subseteq A$.

Hence $A = B$.

[10pts] **(Q6)** A relation R on the set $\mathbb{Z}$ is defined as follows:

$$xRy \iff 4|(x^2 - y^2).$$

(a) Prove that R is an equivalence relation.

(b) Describe [0] and [1], that is, the equivalence classes of 0 and 1, respectively. Use the set-builder notation, and be as explicit as possible.

(c) Show that each $n \in \mathbb{Z}$ is either an element of [0] or an element of [1].

(a) R is reflexive: for all $x \in \mathbb{Z}$,

$x^2 - x^2 = 0 \Rightarrow 4|(x^2 - x^2) \Rightarrow xRx$

R is symmetric: for all $x, y \in \mathbb{Z}$,

$xRy \Rightarrow 4|(x^2 - y^2) \Rightarrow 4|(y^2 - x^2) \Rightarrow yRx$

R is transitive: for all $x, y, z \in \mathbb{Z}$:

$xRy \wedge yRz \Rightarrow 4|(x^2 - y^2) \wedge 4|(y^2 - z^2)$

$\Rightarrow 4|(x^2 - y^2 + y^2 - z^2)$

$\Rightarrow 4|(x^2 - z^2) \Rightarrow xRz$

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

(b) $[0] = \{y \in \mathbb{Z} : 4|(y^2 - 0^2)\} = \{y \in \mathbb{Z} : 4|y^2\}$

$[1] = \{y \in \mathbb{Z} : 4|(y^2 - 1^2)\} = \{y \in \mathbb{Z} : 4|(y^2 - 1)\}$

(c) Take any $n \in \mathbb{Z}$.

Case 1: n is even.

$\Rightarrow n = 2k$ for some $k \in \mathbb{Z}$

$\Rightarrow n^2 = 4k^2$

$\Rightarrow 4 \mid n^2$

$\Rightarrow n \in [0]$ by (b)

<u>Case 2</u>: n is odd.

$\Rightarrow n = 2k+1$ for some $k \in \mathbb{Z}$

$\Rightarrow n^2 = 4(k^2+k)+1$

$\Rightarrow 4 \mid (n^2-1)$

$\Rightarrow n \in [1]$.

[10pts] **(Q7)** (a) Give the full statement for each of the five axioms that we used to define the set of real numbers, $\mathbb{R}$.

(b) Using only the five axioms from (a) and the replacement property, prove the multiplicative cancellation property for real numbers:

For all $x, y, z \in \mathbb{R}$ such that $x \neq 0$, if $xy = xz$, then $y = z$.

Use one axiom per step, and name it, too.

(a) Axiom 7.1 For any $x, y, z \in \mathbb{R}$:

(i) $x+y=y+x$

(ii) $(x+y)+z = x+(y+z)$

(iii) $x(y+z) = xy+xz$

(iv) $xy = yx$

(v) $x(yz) = (xy)z$

Axiom 7.2 There exists $0 \in \mathbb{R}$ s.t. $x+0=x$ for all $x \in \mathbb{R}$.

Axiom 7.3 There exists $1 \in \mathbb{R}$ s.t. $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

Axiom 7.4 For all $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ s.t. $x+y=0$.

Axiom 7.5 For all $x \in \mathbb{R}-\{0\}$ there exists $y \in \mathbb{R}$ s.t. $xy = 1$.

(b) Assume Axioms 7.1–7.5.

Let $x, y, z \in \mathbb{R}$ be such that $xy = xz$ and $x \neq 0$.

By Axiom 7.5, there exists $x^{-1} \in \mathbb{R}$ s.t. $xx^{-1} = 1$.

Then: $xy = xz$

$\Rightarrow \quad x^{-1}(xy) = x^{-1}(xz)$ (replacement)

$\Rightarrow \quad (x^{-1}x)y = (x^{-1}x)z$ (Axiom 7.1(v))

$\Rightarrow \quad (xx^{-1})y = (xx^{-1})z$ (Axiom 7.1(iv))

$\Rightarrow \quad 1 \cdot y = 1 \cdot z$ (Axiom 7.5)

$\Rightarrow \quad y \cdot 1 = z \cdot 1$ (Axiom 7.1(iv))

$\Rightarrow \quad y = z$ (Axiom 7.3)

[10pts] **(Q8)** Let $f : A \longrightarrow B$ be a function.

(a) Give a precise definition that explains what is meant by "f is *injective*".

f is injective if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

(b) Give a precise definition that explains what is meant by "f is *surjective*".

f is surjective if for all $b \in B$, $\exists\, a \in A$ s.t. $f(a) = b$.

(c) Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $f(n) = 3n - 2$.

(i) Prove that f is injective.

(ii) Prove that f is not surjective.

(iii) Find a left inverse g of f.
Be sure to verify that your g is indeed a left inverse of f.

(i) Take any $a_1, a_2 \in \mathbb{Z}$.

$$f(a_1) = f(a_2) \Rightarrow 3a_1 - 2 = 3a_2 - 2$$
$$\Rightarrow 3a_1 = 3a_2$$
$$\Rightarrow a_1 = a_2$$

Hence f is injective.

(ii) Counterexample: let $b = 2 \in \mathbb{Z}$.

Suppose $\exists\, a \in \mathbb{Z}$ s.t. $f(a) = b$

$$\Rightarrow 3a - 2 = 2$$
$$\Rightarrow 3a = 4 \quad \text{, a contradiction.}$$

(iii) If g is a left inverse of f, then $g \circ f = id_{\mathbb{Z}}$

Hence for all $a \in \mathbb{Z}$: $g(f(a)) = a$

$$\Rightarrow g(3a-2) = a$$
$$\Rightarrow g(b) = a \quad \text{for } b = 3a-2$$

Additional work space. Please do not detach.

$$\Rightarrow \quad g(b) = \frac{1}{3}(b+2) \qquad \text{if } b = 3a-2.$$

Define $g : \mathbb{Z} \to \mathbb{Z}$ as follows:

$$g(b) = \begin{cases} \frac{1}{3}(b+2) & \text{if } 3 \mid (b+2) \\ 0 & \text{otherwise} \end{cases}$$

Then for all $a \in \mathbb{Z}$,

$$(g \circ f)(a) = g(f(a)) = g(3a-2) = a$$

so $g \circ f = id_{\mathbb{Z}}$ and g is a left inverse of f.

[10pts] **(Q9)** Let A be a subset of $\mathbb{R}$ defined as

$$A = \left\{ 5 - \frac{2}{n} : n \in \mathbb{N} \right\}.$$

Fully justify all your answers below. In this question, you may use the arithmetic of $\mathbb{R}$ without referring to axioms or propositions.

(a) Find the minimum of A, or else prove that it does not exist.

(b) Find the infimum of A, or else prove that it does not exist.

(c) Find the supremum of A, or else prove that it does not exist.

(a) Claim: $\min(A) = 3$

Proof For all $n \in \mathbb{N}$,

$$n \geq 1$$
$$\Rightarrow \frac{1}{n} \leq 1$$
$$\Rightarrow -\frac{2}{n} \geq -2$$
$$\Rightarrow 5 - \frac{2}{n} \geq 5 - 2 = 3$$

Hence $a \geq 3$ for all $a \in A$, so 3 is a lower bound.
As $5 - \frac{2}{1} = 3 \in A$, we have $\min(A) = 3$.

(b) Since $\min(A) = 3$, we know $\inf(A) = 3$ as well.

(c) Claim: $\sup(A) = 5$

Proof. * For all $n \in \mathbb{N}$,

$$n > 0$$
$$\Rightarrow \frac{1}{n} > 0$$
$$\Rightarrow -\frac{2}{n} < 0$$
$$\Rightarrow 5 - \frac{2}{n} < 5 - 0 = 5$$

Additional work space. Please do not detach.

Hence $a < 5$ for all $a \in A$, so 5 is an upper bound.

* Suppose b is an upper bound for A and $b < 5$.

Since $\mathbb{N}$ is not bounded above, $\exists n \in \mathbb{N}$ s.t. $n > \frac{2}{5-b}$.

$\Rightarrow \quad 5-b > \frac{2}{n}$ (as $5-b>0$)

$\Rightarrow \quad b < 5 - \frac{2}{n}$

$\Rightarrow \quad \exists a \in A$ s.t. $a > b$, a contradiction.

Hence $5 = \sup(A)$.

[10pts] **(Q10)** (a) Let $(a_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$, and $L \in \mathbb{R}$. Give a precise definition that explains what is meant by "the sequence $(a_k)_{k=1}^{\infty}$ *converges* to L".

The sequence $(a_k)_{k=1}^{\infty}$ converges to L if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N} \text{ s.t.})(\forall k \geq N)(|a_k - L| < \varepsilon)$$

(b) Determine $\lim_{k\to\infty} \frac{2k-1}{k+3}$.

You must prove that your answer is correct using the definition of a limit of a sequence from (a), and using no other results on limits.

Claim $\lim_{k\to\infty} \frac{2k-1}{k+3} = 2$

Proof. * Take any $\varepsilon > 0$. We want $N \in \mathbb{N}$ s.t. $\forall k \geq N$, $\left|\frac{2k-1}{k+3} - 2\right| < \varepsilon$.

* Calculation:

$$\left|\frac{2k+1}{k+3} - 2\right| = \left|\frac{2k+1-2(k+3)}{k+3}\right| = \left|\frac{-5}{k+3}\right| = \frac{5}{k+3} < \frac{5}{k}$$

For $\left|\frac{2k+1}{k+3} - 2\right| < \varepsilon$, it suffices that $\frac{5}{k} < \varepsilon$, i.e. $k > \frac{5}{\varepsilon}$

* Choose N: Since $\mathbb{N}$ is not bounded above, $\exists N \in \mathbb{N}$ s.t. $N > \frac{5}{\varepsilon}$.

* Verify this N: for all $k \geq N$,

$$\left|\frac{2k+1}{k+3} - 2\right| = \frac{5}{k+3} < \frac{5}{k} \leq \frac{5}{N} < \varepsilon.$$

Hence $\lim_{k\to\infty} \frac{2k+1}{k+3} = 2$.

[10pts]**(Q11)** Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ defined recursively as follows:

$$x_1 = 1, \quad \text{and} \quad x_n = \frac{1}{3}(x_{n-1} + 6) \quad \text{for all } n \geq 2.$$

(a) Using induction, prove that this sequence is bounded above by 3 and bounded below by 0; that is, prove that for all $n \in \mathbb{N}$,

$$0 \leq x_n \leq 3.$$

(b) Prove that this sequence is increasing.

(c) **(Bonus)** Is this sequence convergent? Justify your answer.

(a) Let $P(n)$: "$0 \le x_n \le 3$"

BI: To prove $P(1)$: "$0 \le x_1 \le 3$"

Since $x_1 = 1$ and $0 \le 1 \le 3$, $P(1)$ is T.

IS: To prove $P(n) \Rightarrow P(n+1)$ for all $n \geq 1$.

Fix any $n \geq 1$. Assume $P(n)$: "$0 \le x_n \le 3$" (IH)

Examine $P(n+1)$: "$0 \le x_{n+1} \le 3$"

$$x_{n+1} = \frac{1}{3}(x_n + 6) = \frac{1}{3}x_n + 2 \overset{IH}{\le} \frac{1}{3}\cdot 3 + 2 = 3$$

and $x_{n+1} = \frac{1}{3}(x_n + 6) \geq \frac{1}{3}(0+6) = 2 \geq 0$

so $0 \le x_{n+1} \le 3$, and $P(n+1)$ follows.

By induction, $P(n)$ is T for all $n \geq 1$.

(b) For all $n \in \mathbb{N}$:

$$x_{n+1} - x_n = \frac{1}{3}(x_n + 6) - x_n = -\frac{2}{3}x_n + 2 \geq -\frac{2}{3}\cdot 3 + 2 = 0$$

since $x_n \le 3$ by (a).

Additional work space. Please do not detach.

Hence $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, and $(x_n)_{n=1}^{\infty}$ is increasing.

(c) By (a), the sequence is bounded, and by (b), it is monotonic. Since every monotonic bounded sequence converges, $(x_n)_{n=1}^{\infty}$ converges.

Table of Logical Equivalences

	Equivalence	Name
(1)	$P \Rightarrow Q \equiv \neg P \vee Q$	Implication Law
(2)	$P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	Biconditional Laws
(3)	$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$	
(4)	$P \vee \neg P \equiv \mathbf{T}$	Negation Laws
(5)	$P \wedge \neg P \equiv \mathbf{F}$	
(6)	$P \vee \mathbf{F} \equiv P$	Identity Laws
(7)	$P \wedge \mathbf{T} \equiv P$	
(8)	$P \vee \mathbf{T} \equiv \mathbf{T}$	Domination Laws
(9)	$P \wedge \mathbf{F} \equiv \mathbf{F}$	
(10)	$P \vee P \equiv P$	Idempotent Laws
(11)	$P \wedge P \equiv P$	
(12)	$\neg\neg P \equiv P$	Double negation
(13)	$P \vee Q \equiv Q \vee P$	Commutative Laws
(14)	$P \wedge Q \equiv Q \wedge P$	
(15)	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	Associative Laws
(16)	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
(17)	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	Distributive Laws
(18)	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	
(19)	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	De Morgan's Laws
(20)	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	