MAT1362
Winter 2023
Midterm 2
Prof. Antoine Poirier

You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is an 75-minute **closed-book** exam; no notes are allowed. **Calculators and notes are not permitted.**
- The exam consists of 5 questions, with a maximum of 40 points. If you need more additional space, you can use the backs of any of the pages. **Do not detach any pages**.
- Question 1 comprises ten true or false questions worth 1 point each. Circle the correct answer. There is no penalty for an incorrect answer.
- Questions 2–5 are long-answer questions worth points as indicated. You must show all relevant steps and clearly justify your answers in order to obtain full marks.
- **Cellular phones** and other electronic devices **are not permitted** during this exam. Phones and other devices must be turned off completely and stored out of reach. Do not keep them in your possession, such as in your pockets. If you are caught with such a device, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

LAST NAME:_____

First name:_____

Student Number:_____

Signature:_____

1. **(10 pts)** For each of the following statements, determine whether it is true or false, and circle the correct answer. No justification is necessary.

   (a) If $A \subseteq \mathbb{R}$ and A has a supremum, then A has a maximum. **True** **False**

   (b) If $A \subseteq \mathbb{R}$ and A has a minimum, then A has an infimum. **True** **False**

   (c) If A and B are subsets of X, then $(A \cup B) \cap (A \cup B^C) = A$. **True** **False**

   (d) $\forall x, y \in \mathbb{Z}.(x < y \implies \exists z \in \mathbb{R}.x < z < y)$. **True** **False**

   (e) If p is a prime number and $k \in \mathbb{Z}$, then $k(k^{p-1} - 1)$ is divisible by p. **True** **False**

   (f) The empty relation $\emptyset \subseteq \mathbb{Z} \times \mathbb{Z}$ is symmetric. **True** **False**

   (g) $\forall x \in \mathbb{R}.(x \neq 0 \implies \exists y \in \mathbb{R}, xy = 1)$. **True** **False**

   (h) $\{\{1, 3, 5\}, \{2, 4\}, \{1, 2\}\}$ is a partition of the set $A = \{1, 2, 3, 4, 5\}$. **True** **False**

   (i) In $\mathbb{Z}_8$, we have $[3] \odot [5] = [1]$. **True** **False**

   (j) If $A = \emptyset$ or $B = \emptyset$, then $A \times B = B \times A$. **True** **False**

2. **(6pts)** Consider the following sets:

$$A = \{8n - 1 \mid n \in \mathbb{N}\}, \qquad B = \{4m + 3 \mid m \in \mathbb{N}\}$$

(a) Show that $A \subseteq B$.

(b) Show that B is not a subset of A.

3. **(8pts)** Consider the relation $\sim$ defined on the set $\mathbb{Z} \times \mathbb{N}$ given by

$$(x, y) \sim (a, b) \iff xb = ay.$$

(a) Sow that $\sim$ is an equivalence relation.

(b) Give a list of three members from each of the equivalence classes of $(-1, 1)$ and $(1, 1)$.

4. **(8pts)**

(a) What is the remainder of 3^{45} when divided by 7?

(b) What is the additive inverse of [27] in $\mathbb{Z}_7$?

(c) Does [27] have a multiplicative inverse in $\mathbb{Z}_7$? If so, find it. That is, is there some integer $k \in \mathbb{Z}$ such that $[27] \odot [k] = [1]$ in $\mathbb{Z}_7$?

5. **(8pts)** Consider the set

$$E = \left\{ 4 - \frac{2}{n} \middle| n \in \mathbb{N} \right\}.$$

That is,

$$E = \left\{ x \in \mathbb{R} \middle| \exists n \in \mathbb{N} \text{ such that } x = 4 - \frac{2}{n} \right\}.$$

(a) Show directly that E admits a supremum and an infimum.

(b) Show that 4 is the supremum of E.

(c) Show that 2 is the minimum of E. What is the infimum of E?

(d) Is 4 the maximum of E?