**MAT1362**
**Winter 2023**
**Midterm 2**
Prof. Antoine Poirier

You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is an 75-minute **closed-book** exam; no notes are allowed. **Calculators and notes are not permitted.**
- The exam consists of 5 questions, with a maximum of 40 points. If you need more additional space, you can use the backs of any of the pages. **Do not detach any pages**.
- Question 1 comprises ten true or false questions worth 1 point each. Circle the correct answer. There is no penalty for an incorrect answer.
- Questions 2–5 are long-answer questions worth points as indicated. You must show all relevant steps and clearly justify your answers in order to obtain full marks.
- **Cellular phones** and other electronic devices **are not permitted** during this exam. Phones and other devices must be turned off completely and stored out of reach. Do not keep them in your possession, such as in your pockets. If you are caught with such a device, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

LAST NAME:____________________

First name:____________________

Student Number:____________________

Signature:____________________

1. **(10 pts)** For each of the following statements, determine whether it is true or false, and circle the correct answer. No justification is necessary.

   (a) If $A \subseteq \mathbb{R}$ and $A$ has a supremum, then $A$ has a maximum. **True** **False**

   (b) If $A \subseteq \mathbb{R}$ and $A$ has a minimum, then $A$ has an infimum. **True** **False**

   (c) If $A$ and $B$ are subsets of $X$, then $(A \cup B) \cap (A \cup B^C) = A$. **True** **False**

   (d) $\forall x, y \in \mathbb{Z}.(x < y \implies \exists z \in \mathbb{R}. x < z < y)$. **True** **False**

   (e) If $p$ is a prime number and $k \in \mathbb{Z}$, then $k(k^{p-1} - 1)$ is divisible by $p$. **True** **False**

   (f) The empty relation $\emptyset \subseteq \mathbb{Z} \times \mathbb{Z}$ is symmetric. **True** **False**

   (g) $\forall x \in \mathbb{R}.(x \neq 0 \implies \exists y \in \mathbb{R}, xy = 1)$. **True** **False**

   (h) $\{\{1, 3, 5\}, \{2, 4\}, \{1, 2\}\}$ is a partition of the set $A = \{1, 2, 3, 4, 5\}$. **True** **False**

   (i) In $\mathbb{Z}_8$, we have $[3] \odot [5] = [1]$. **True** **False**

   (j) If $A = \emptyset$ or $B = \emptyset$, then $A \times B = B \times A$. **True** **False**

**Solutions:**

(a) False. The open interval $(0, 1)$ has a supremum of 1, but no maximum.
(b) True. The infimum is equal to the minimum if the minimum exists.
(c) True. $(A \cup B) \cap (A \cup B^C) = A \cap (B \cup B^C) = A \cap \mathcal{U} = A$.
(d) True. Between any two integers, there is a real number.
(e) True. Either $p$ divides $k$, in which case it divides $k(k^{p-1} - 1)$, or the corollary of Fermat's little theorem apply, in which case $p$ divides $k^{p-1} - 1$.
(f) True. The implication $(x, y) \in \emptyset \implies (y, x) \in \emptyset$ is true because $(x, y) \in \emptyset$ is false.
(g) True. Every non-zero real number has a multiplicative inverse.
(h) False. $\{2, 4\}$ and $\{1, 2\}$ are not disjoint.
(i) False. $[3] \odot [5] = [15] = [7] \neq [1]$.
(j) True. In this case, $A \times B = B \times A = \emptyset$.

2. **(6pts)** Consider the following sets:

$$A = \{8n - 1 \mid n \in \mathbb{N}\}, \qquad B = \{4m + 3 \mid m \in \mathbb{N}\}$$

(a) Show that $A \subseteq B$.

**Solution:**

Let $x \in A$. There exists $n \in \mathbb{N}$ such that $x = 8n - 1$. We can rewrite $x = 8n - 1 = 8n - 4 + 3 = 4(2n - 1) + 3$. Then set $m = 2n - 1$. Since $n \in \mathbb{N}$, then $m \in \mathbb{N}$ as well. This means $x = 4m + 3$ where $m \in \mathbb{N}$. So $x \in B$. This shows $A \subseteq B$.

(b) Show that $B$ is not a subset of $A$.

**Solution:**

We have $11 \in B$ but $11 \notin A$. We have $11 \in B$ because when $m = 2$, we get $4m + 3 = 11$. However, if $11 \in A$, then there exists $n \in \mathbb{N}$ such that $8n - 1 = 11$, which simplifies to $2n = 3$. This means 3 is even, which is a contradiction. We conclude $11 \notin A$.

3. **(8pts)** Consider the relation $\sim$ defined on the set $\mathbb{Z}\times\mathbb{N}$ given by

$$(x,y)\sim(a,b) \iff xb = ay.$$

(a) Sow that $\sim$ is an equivalence relation.

**Solution:** We have $(x,y)\sim(a,b) \iff \frac{x}{y}=\frac{a}{b}$.

This relation is reflexive: for all $(x,y)\in\mathbb{Z}\times\mathbb{N}$, we have $(x,y)\sim(x,y)$ because $\frac{x}{y}=\frac{x}{y}$.

This relation is symmetric: suppose $(x,y)\sim(a,b)$, then $\frac{x}{y}=\frac{a}{b}$. But since equality is symmetric, we get $\frac{a}{b}=\frac{x}{y}$, implying that $(a,b)\sim(x,y)$.

This relation is transitive: suppose $(x,y)\sim(a,b)$ and $(a,b)\sim(u,v)$. Then $\frac{x}{y}=\frac{a}{b}$ and $\frac{a}{b}=\frac{u}{v}$. Since equality is transitive, we get $\frac{x}{y}=\frac{u}{v}$, implying that $(x,y)\sim(u,v)$

Because $\sim$ is reflexive, symmetric and transitive, it is an equivalence relation.

(b) Give a list of three members from each of the equivalence classes of $(-1,1)$ and $(1,1)$.

**Solution:**

Three members of $[(-1,1)]$ are $(-1,1),(-2,2)$ and $(-3,3)$. Three members of $(1,1)$ are $(1,1),(2,2)$ and $(3,3)$.

4. **(8pts)**

(a) What is the remainder of $3^{45}$ when divided by 7?

**Solution:** We first note that 7 does not divide 3, so $3^6 \equiv 1 \mod 7$.

We then get $3^{45} = (3^6)^7 \cdot 3^3 \equiv 1^7 \cdot 3^3 \equiv 3^3 \equiv 27 \equiv 6 \mod 7$

(b) What is the additive inverse of $[27]$ in $\mathbb{Z}_7$?

**Solution:**

First, $27 \equiv 6 \mod 7$, and since $6 + 1 \equiv 0 \mod 7$, we conclude that the additive inverse of $[27]$ is $[1]$.

(c) Does $[27]$ have a multiplicative inverse in $\mathbb{Z}_7$? If so, find it. That is, is there some integer $k \in \mathbb{Z}$ such that $[27] \odot [k] = [1]$ in $\mathbb{Z}_7$?

**Solution:**

Yes, $[27] = [6]$, and $[6] \odot [6] = [36] = [1]$, so $[6]$ is the multiplicative inverse of $[27]$ in $\mathbb{Z}_7$.

5. **(8pts)** Consider the set
$$E = \left\{4 - \frac{2}{n} \middle| n \in \mathbb{N}\right\}.$$
That is,
$$E = \left\{x \in \mathbb{R} \middle| \exists n \in \mathbb{N} \text{ such that } x = 4 - \frac{2}{n}\right\}.$$

(a) Show directly that $E$ admits a supremum and an infimum.

**Solution:**

Since $\frac{2}{n} > 0$, we have $4 - \frac{2}{n} < 4$ for all $n \in \mathbb{N}$, implying that $E$ is bounded above by 4. Similarly, since $n \geq 1$, we get $\frac{2}{n} \leq 2$, hence $4 - \frac{2}{n} \geq 2$ for all $n \in \mathbb{N}$, implying that $E$ is bounded below by 2. Since $E$ is bounded above, it has a supremum. Since $E$ is bounded below, it has an infimum.

(b) Show that 4 is the supremum of $E$.

**Solution:**

In part (a), we have proved that 4 is an upper bound on $E$. We now show it is the least upper bound. Suppose $b < 4$ is an upper bound on $E$, this implies, for all $n \in \mathbb{N}$,

$$\begin{aligned}
& b \geq 4 - \frac{2}{n} \\
\implies & \frac{2}{n} \geq 4 - b \\
\implies & \frac{n}{2} \leq \frac{1}{4-b} && \left(\text{Since } 4 - b > 0 \text{ and } \frac{2}{n} > 0\right) \\
\implies & n \leq \frac{2}{4-b}
\end{aligned}$$

The conclusion to these equations is that every natural number is smaller than $\frac{2}{4-b}$, which is a contradiction. Therefore, $b$ is not an upper bound on $E$. We conclude $\sup(E) = 4$.

(c) Show that 2 is the minimum of $E$. What is the infimum of $E$?

**Solution:**

In part (a), we have established that 2 is a lower bound on $E$. Furthermore, $2 \in E$ (choose $n = 1$). These two observation combined shows that $\min(E) = 2$. Since the minimum exists, we conclude that $\inf(E) = 2$ as well.

(d) Is 4 the maximum of $E$?

**Solution:**

4 is not the maximum, because $4 \notin E$. To show this, we note that since $n > 0$, we get $\frac{2}{n} > 0$ for all $n \in \mathbb{N}$. This means $4 - \frac{2}{n} < 4$ for all $n \in \mathbb{N}$, hence $4 - \frac{2}{n} \neq 4$ for all $n \in \mathbb{N}$.