MAT1362
Winter 2023
Midterm 1
Prof. Antoine Poirier

You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is an 75-minute **closed-book** exam; no notes are allowed. **Calculators and notes are not permitted.**
- The exam consists of 5 questions, with a maximum of 35 points. If you need more additional space, you can use the backs of any of the pages. **Do not detach any pages**.
- Question 1 comprises ten true or false questions worth 1 point each. Circle the correct answer. There is no penalty for an incorrect answer.
- Questions 2–5 are long-answer questions worth points as indicated. You must show all relevant steps and clearly justify your answers in order to obtain full marks.
- **Cellular phones** and other electronic devices **are not permitted** during this exam. Phones and other devices must be turned off completely and stored out of reach. Do not keep them in your possession, such as in your pockets. If you are caught with such a device, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

LAST NAME:____________________

First name:____________________

Student Number:____________________

Signature:____________________

1. **(10 pts)** For each of the following statements, determine whether it is true or false, and circle the correct answer. No justification is necessary.

   (a) $\exists x \in \mathbb{Z}, x > 0.$ **True** **False**

   (b) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 1.$ **True** **False**

   (c) $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, x + y = 1.$ **True** **False**

   (d) $\exists a \in \mathbb{Z}, \forall b \in \mathbb{Z}, ab = 0.$ **True** **False**

   (e) $\exists! x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy = y.$ **True** **False**

   (f) $\forall y \in \mathbb{Z}, \exists! x \in \mathbb{Z}, xy = y.$ **True** **False**

   (g) $(1 < 0) \Rightarrow (1 = 0).$ **True** **False**

   (h) $\exists x \in \mathbb{Z}, ((x = 1) \Rightarrow (x^2 = 2)).$ **True** **False**

   (i) $\exists a \in \mathbb{Z}, a < 0$ and $a > 0.$ **True** **False**

   (j) $(\exists a \in \mathbb{Z}, a < 0)$ and $(\exists a \in \mathbb{Z}, a > 0).$ **True** **False**

2. **(6pts)** Prove the following statements using only axioms of $\mathbb{Z}$ (commutativity, associativity, distributivity, additive and multiplicative identities, additive inverse, cancellation), replacement, and the definition of $x - y$. At each step, specify which axiom you used.

   (a) $\forall n \in \mathbb{Z}, n \cdot 0 = 0$.

   (b) $\forall a, b, c \in \mathbb{Z}$, if $a \neq 0$ and $a(x + b) = a$, then $x = 1 - b$.

3. **(7 pts)** Consider the following implication:

$$a|b \text{ and } a|c \Rightarrow a|(b+c).$$

(a) Write the converse.

(b) Write the contrapositive.

(c) Prove that the implication is true.

(d) Is the contrapositive true for all a, b, and c? (Justify)

(e) Is the converse implication true for all a, b, and c? (Justify)

4. **(6 pts)** Prove by induction that

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 1.$$

5. **(6 pts)** Consider the sequence (called the Fibonacci sequence) defined by

$$F_{n+2} = F_{n+1} + F_n,$$
$$F_0 = F_1 = 1.$$

Prove that $F_n \leq 2^n$ for all $n \geq 0$.