MAT1362
Winter 2023
Midterm 1

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You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is an 75-minute **closed-book** exam; no notes are allowed. **Calculators and notes are not permitted.**
- The exam consists of 5 questions, with a maximum of 35 points. If you need more additional space, you can use the backs of any of the pages. **Do not detach any pages**.
- Question 1 comprises ten true or false questions worth 1 point each. Circle the correct answer. There is no penalty for an incorrect answer.
- Questions 2–5 are long-answer questions worth points as indicated. You must show all relevant steps and clearly justify your answers in order to obtain full marks.
- **Cellular phones** and other electronic devices **are not permitted** during this exam. Phones and other devices must be turned off completely and stored out of reach. Do not keep them in your possession, such as in your pockets. If you are caught with such a device, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

LAST NAME:____________________________

First name:____________________________

Student Number:____________________________

Signature:____________________________

1. **(10 pts)** For each of the following statements, determine whether it is true or false, and circle the correct answer. No justification is necessary.

   (a) $\exists x \in \mathbb{Z}, x > 0.$ **True** **False**

   (b) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 1.$ **True** **False**

   (c) $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, x + y = 1.$ **True** **False**

   (d) $\exists a \in \mathbb{Z}, \forall b \in \mathbb{Z}, ab = 0.$ **True** **False**

   (e) $\exists! x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy = y.$ **True** **False**

   (f) $\forall y \in \mathbb{Z}, \exists! x \in \mathbb{Z}, xy = y.$ **True** **False**

   (g) $(1 < 0) \Rightarrow (1 = 0).$ **True** **False**

   (h) $\exists x \in \mathbb{Z}, ((x = 1) \Rightarrow (x^2 = 2)).$ **True** **False**

   (i) $\exists a \in \mathbb{Z}, a < 0$ and $a > 0.$ **True** **False**

   (j) $(\exists a \in \mathbb{Z}, a < 0)$ and $(\exists a \in \mathbb{Z}, a > 0).$ **True** **False**

   Solutions:

   (a) True. $1 > 0$.
   (b) False.
   (c) True. $x = (-y) + 1$ works.
   (d) True. Try $a = 0$. Then $a \cdot b = 0$ for all b.
   (e) True. This says that x is a multiplicative identity, and we know these are unique.
   (f) False. For $y = 0$, we can take $2 \cdot 0 = 1 \cdot 0 = 0$.
   (g) True. $F \implies F$ is true.
   (h) True. Take $x = 2$. Then $F \implies F$ is true.
   (i) False. No such a exists, you must be > 0, < 0 or $= 0$.
   (j) True. One can take $a = -1$ to satisfy the statement on the left, and $a = 1$ to satisfy the statement on the right.

2. **(6pts)** Prove the following statements using only axioms of $\mathbb{Z}$ (commutativity, associativity, distributivity, additive and multiplicative identities, additive inverse, cancellation), replacement, and the definition of $x - y$. At each step, specify which axiom you used.

(a) $\forall n \in \mathbb{Z}, n \cdot 0 = 0$.

Solutions: Let $n \in \mathbb{Z}$. Then

$$\begin{aligned}
& 0 = 0 + 0 && \text{(additive identity)}\\
\Rightarrow\ & n \cdot 0 = n \cdot (0 + 0) && \text{(replacement)}\\
\Rightarrow\ & n \cdot 0 = n \cdot 0 + n \cdot 0 && \text{(distributivity)}\\
\Rightarrow\ & n \cdot 0 + (-(n \cdot 0)) = (n \cdot 0 + n \cdot 0) + (-(n \cdot 0)) && \text{(replacement)}\\
\Rightarrow\ & 0 = (n \cdot 0 + n \cdot 0) + (-(n \cdot 0)) && \text{(additive inverse)}\\
\Rightarrow\ & 0 = n \cdot 0 + (n \cdot 0 + (-(n \cdot 0))) && \text{(associativity of +)}\\
\Rightarrow\ & 0 = n \cdot 0 + 0 && \text{(additive inverse)}\\
\Rightarrow\ & 0 = n \cdot 0. && \text{(additive identity)}
\end{aligned}$$

(b) $\forall a, b, c \in \mathbb{Z}$, if $a \neq 0$ and $a(x + b) = a$, then $x = 1 - b$.

Solutions:

Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $a(x + b) = a$.

$$\begin{aligned}
& a(x + b) = a && \\
\iff\ & a(x + b) = a \cdot 1 && \text{(multiplicative identity)}\\
\iff\ & x + b = 1 && \text{(cancellation since } a \neq 0)\\
\iff\ & (x + b) + (-b) = 1 + (-b) && \text{(replacement)}\\
\iff\ & x + (b + (-b)) = 1 + (-b) && \text{(associativity of +)}\\
\iff\ & x + 0 = 1 + (-b) && \text{(additive inverse)}\\
\iff\ & x = 1 + (-b) && \text{(additive inverse)}\\
\iff\ & x = 1 - b. && \text{(definition of subtraction)}
\end{aligned}$$

3. **(7 pts)** Consider the following implication:

$$a|b \text{ and } a|c \Rightarrow a|(b+c).$$

(a) Write the converse.

Solutions:

The converse is

$$a|(b+c) \Rightarrow a|b \text{ and } a|c.$$

(b) Write the contrapositive.

Solutions:

The contrapositive is

$$a \nmid (b+c) \Rightarrow a \nmid b \text{ or } a \nmid c.$$

(c) Prove that the implication is true.

Solutions:

Let $a, b, c \in \mathbb{Z}$ be such that $a|b$ and $a|c$. Thus there are $j, k \in \mathbb{Z}$ such that $b = ja$ and $c = ka$. But then

$$b + c = (ja) + (ka) = (j+k)a.$$

Since $j + k \in \mathbb{Z}$, it follows that $a|b + c$.

(d) Is the contrapositive true for all a, b, and c? (Justify)

Solutions:

The contrapositive is true since the contrapositive is equivalent to the implication, which is true.

(e) Is the converse implication true for all a, b, and c? (Justify)

Solutions:

The converse is false. Take $b = c = 1$ and $a = 2$. Then $a|b + c$, but a divides neither b nor c.

4. **(6 pts)** Prove by induction that

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2.$$

Solutions:

Let $P(n)$ be the statement that $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$.

Base case We have

$$\sum_{k=1}^{1} k2^k = 1 \cdot 2^1 = 2 = (1-1) \cdot 2^{1+1} + 2.$$

So $P(1)$ is true.

Induction step. Suppose that $n \in \mathbb{N}$ and that $P(n)$ holds. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k2^k &= \sum_{k=1}^{n} k2^k + (n+1)2^{n+1} \\
&= (n-1)2^{n+1} + 2 + (n+1)2^{n+1} && \text{(Induction hypothesis)} \\
&= n2^{n+1} - 2^{n+1} + 2 + n2^{n+1} + 2^{n+1} \\
&= 2n2^{n+1} + 2 \\
&= n2^{n+2} + 2 \\
&= ((n+1)-1)2^{((n+1)+1)} + 2.
\end{aligned}$$

5. **(6 pts)** Consider the sequence (called the Fibonacci sequence) defined by

$$F_{n+2} = F_{n+1} + F_n,$$
$$F_0 = F_1 = 1.$$

Prove that $F_n \leq 2^n$ for all $n \geq 0$.

Solutions

We prove this by induction. Let $P(n)$ be the statement that $F_n \leq 2^n$.

Base case. We check this holds for F_0, F_1. Well since $F_0 = 1 \leq 1 = 2^0$, this is true, and since $F_1 = 1 \leq 2 = 2^1$, this is true.

Induction step. Suppose that $n \geq 1$ and that $P(k)$ holds for $k = 0, \ldots, n$. Then

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\leq 2^n + F_{n-1} && \text{(Induction hypothesis)} \\ &\leq 2^n + 2^{n-1} && \text{(Induction hypothesis)} \\ &\leq 2^n + 2^n && \text{(since } 2^{n-1} \leq 2^n\text{)} \\ &= 2 \cdot 2^n \\ &= 2^{n+1}. \end{aligned}$$

Thus $P(n+1)$ holds. Therefore by the principle of strong induction, $P(n)$ holds for all $n \geq 0$.