MAT1362
Winter 2023
Final Exam
April 23, 2023
Instructor: Antoine Poirier

You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is an 180-minute **closed-book** exam; no notes are allowed, except for one cheat-sheet. **Calculators and other notes are not permitted.**
- The exam consists of 9 questions, with a maximum of 58 points. If you need additional space, you can use the backs of any of the pages. **Do not detach any pages**.
- Question 1 comprises ten true or false questions worth 1 point each. Circle the correct answer. There is no penalty for an incorrect answer.
- Questions 2–9 are long-answer questions worth points as indicated. You must show all relevant steps and clearly justify your answers in order to obtain full marks.
- **Cellular phones** and other electronic devices **are not permitted** during this exam. Phones and other devices must be turned off completely and stored out of reach. Do not keep them in your possession, such as in your pockets. If you are caught with such a device, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

LAST NAME:____________________

First name:____________________

Student Number:____________________

Seat number:____________________

Signature:____________________

1. **(10 pts)** For each of the following statements, determine whether it is true or false, and circle the correct answer. No justification is necessary.

   (a) If $A \subseteq \mathbb{R}$ has an infimum, then A has a minimum. **True** **False**

   (b) The set $\{(1,1),(1,3),(3,1),(3,3),(2,4),(4,2),(2,2),(4,4)\}$ is an equivalence relation on $\{1,2,3,4\}$. **True** **False**

   (c) Let $a, b, c \in \mathbb{N}$. Then, if c divides neither a nor b, then c does not divide ab. **True** **False**

   (d) The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$ is bijective. **True** **False**

   (e) The function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ defined by $f(x) = x^2$ is bijective. **True** **False**

   (f) Let x be an integer. Then $x^4 \equiv 1 \mod 5$ if and only if 5 does not divide x. **True** **False**

   (g) Let $f : X \to Y$ be an injective function. Then, f has a unique left inverse. **True** **False**

   (h) A bounded sequence cannot diverge. **True** **False**

   (i) If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are divergent sequences, then $(a_n + b_n)_{n=1}^{\infty}$ also diverges. **True** **False**

   (j) If $(a_n)_{n=1}^{\infty}$ is a convergent sequence and $(b_n)_{n=1}^{\infty}$ is a divergent sequence, then $(a_n + b_n)_{n=1}^{\infty}$ diverges. **True** **False**

2. **(6 pts)** Consider the following proposition P:

$$\forall x \in A, \exists y \in A \text{ such that } (x < y \wedge \forall z \in A, (x < z \implies y \leq z))$$

(a) **(2 pts)** Write the negation of P. Simplify your answer so it does not contain the negation symbol "$\neg$".

(b) **(2 pts)** Is P true when $A = \mathbb{Z}$? Justify your answer.

(c) **(2 pts)** Is P true when $A = \mathbb{R}$? Justify your answer.

3. **(6 pts)** Show that for all $n \geq 1$,

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

4. **(6 pts)** Consider the following subsets of $\mathbb{Z}$:

$$A = \{x \in \mathbb{Z} \mid x^2 + 2x > 0\}, \quad B = \{x \in \mathbb{Z} \mid |x+1| > 1\}$$

Show that $A = B$.

5. **(6 pts)**

   (a) **(1 pt)** Give the definition of an equivalence relation on a set X. If you use words like "transitive", briefly define them.

   (b) **(3 pts)** Show that the relation on $\mathbb{R}$ defined by

   $$x \sim y \iff \exists q \in \mathbb{Z} \text{ such that } y = x + q$$

   is an equivalence relation.

(c) **(2 pts)** Show that for any $x \in \mathbb{R}$, there exists a real number $y \in [0, 1)$ such that $[x]_\sim = [y]_\sim$, where $\sim$ is the relation from (b).

6. **(6 pts)**

   (a) **(3 pts)** What are the solutions $x \in \mathbb{Z}$ of the equation

   $$x^2 + x + 1 \equiv 2 \mod 5?$$

   Hint: you can express the set of solutions in terms of the possible remainders of x upon division by 5.

   (b) **(3 pts)** What is the last digit of 3^{33}? That is, calculate the remainder upon division by 10.

7. **(5 pts)** Consider the subset S of $\mathbb{R}$ given as follows:

$$S := \left\{ 2 + \frac{1}{3x} \mid x \in \mathbb{R} \text{ and } x \geq 2 \right\}.$$

(a) **(2 pts)** Find the maximum of S, with justification, or prove that it does not exist.

(b) **(1 pt)** Find the supremum of S, with justification, or prove that it does not exist.

(c) **(2 pts)** Find the infimum of S, with justification, or prove that it does not exist.

8. **(6 pts)** Consider the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{2x} - 1.$$

Justify your answers to the following questions.

(a) **(2 pts)** Is f injective?

(b) **(2 pts)** What is the image of f? Use interval notation in your final answer.

(c) **(1 pt)** Is f bijective?

(d) **(1 pt)** Does f have a left or right inverse?

9. **(7 pts)**

(a) **(1 pt)** State what it means for a sequence $(x_n)_{n=1}^{\infty}$ of real numbers to converge to $L \in \mathbb{R}$.

(b) **(3 pts)** Prove, using the definition of a limit, that

$$\lim_{n\to\infty}\left(13+\frac{2}{4+n^2}\right)=13.$$

(c) **(3 pts)** Consider the sequence $(x_n)_{n=1}^{\infty}$ defined recursively by $x_1 = 3$ and, for $n \geq 1$,

$$x_{n+1} = \begin{cases} x_n + 1 & \text{if } n \text{ is even} \\ x_n - 1 & \text{if } n \text{ is odd} \end{cases}$$

Does the sequence $(x_n)_{n=1}^{\infty}$ converge or diverge? Justify your answer.