MAT1362
Winter 2023
Final Exam
April 23, 2023
Instructor: Antoine Poirier

You must **sign below** to confirm that you have read, understand, and will follow these **instructions**:

- This is an 180-minute **closed-book** exam; no notes are allowed, except for one cheat-sheet. **Calculators and other notes are not permitted.**
- The exam consists of 9 questions, with a maximum of 58 points. If you need additional space, you can use the backs of any of the pages. **Do not detach any pages**.
- Question 1 comprises ten true or false questions worth 1 point each. Circle the correct answer. There is no penalty for an incorrect answer.
- Questions 2–9 are long-answer questions worth points as indicated. You must show all relevant steps and clearly justify your answers in order to obtain full marks.
- **Cellular phones** and other electronic devices **are not permitted** during this exam. Phones and other devices must be turned off completely and stored out of reach. Do not keep them in your possession, such as in your pockets. If you are caught with such a device, the following may occur: academic fraud allegations will be filed which may result in your obtaining a 0 (zero) for the exam.

LAST NAME:____________________

First name:____________________

Student Number:____________________

Seat number:____________________

Signature:____________________

1. **(10 pts)** For each of the following statements, determine whether it is true or false, and circle the correct answer. No justification is necessary.

   (a) If $A \subseteq \mathbb{R}$ has an infimum, then A has a minimum. **False**

   (b) The set $\{(1,1),(1,3),(3,1),(3,3),(2,4),(4,2),(2,2),(4,4)\}$ is an equivalence relation on $\{1,2,3,4\}$. **True**

   (c) Let $a, b, c \in \mathbb{N}$. Then, if c divides neither a nor b, then c does not divide ab. **False**

   (d) The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$ is bijective. **False**

   (e) The function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ defined by $f(x) = x^2$ is bijective. **True**

   (f) Let x be an integer. Then $x^4 \equiv 1 \mod 5$ if and only if 5 does not divide x. **True**

   (g) Let $f : X \to Y$ be an injective function. Then, f has a unique left inverse. **False**

   (h) A bounded sequence cannot diverge. **False**

   (i) If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are divergent sequences, then $(a_n + b_n)_{n=1}^{\infty}$ also diverges. **False**

   (j) If $(a_n)_{n=1}^{\infty}$ is a convergent sequence and $(b_n)_{n=1}^{\infty}$ is a divergent sequence, then $(a_n + b_n)_{n=1}^{\infty}$ diverges. **True**

2. **(6 pts)** Consider the following proposition P:

$$\forall x \in A, \exists y \in A \text{ such that } (x < y \wedge \forall z \in A, (x < z \implies y \leq z))$$

(a) **(2 pts)** Write the negation of P. Simplify your answer so it does not contain the negation symbol "$\neg$".

$$\exists x \in A. \forall y \in A. (x \geq y \vee \exists z \in A. (x < z \wedge y > z))$$

(b) **(2 pts)** Is P true when $A = \mathbb{Z}$? Justify your answer.

This is true when $A = \mathbb{Z}$. Let $x \in \mathbb{Z}$ and take $y = x + 1$. Then $x < y$ is true, since $x < x + 1$ is true. Also, for all $z \in \mathbb{Z}$, if $x < z$, then z is at least $x + 1$, hence $y \leq z$.

(c) **(2 pts)** Is P true when $A = \mathbb{R}$? Justify your answer.

This is false when $A = \mathbb{R}$. We prove the negation is true. Take $x = 0$ and consider $y \in \mathbb{R}$. Now, if $y \notin \mathbb{R}_{>0}$, then "$x \geq y$" is true, hence $\neg P$ is true. Otherwise, if $y \in \mathbb{R}_{>0}$, we must find $z \in \mathbb{R}$ such that $x < z < y$. Such a z exists because $\mathbb{R}$ is dense. Hence $\neg P$ is true.

3. **(6 pts)** Show that for all $n \geq 1$,

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

We use induction on n. For all $n \geq 1$. Let $P(n)$ be the statement "$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$".

Base case: $P(1)$ says "$\sum_{i=1}^{1} i(i+1) = \frac{1(1+1)(1+2)}{3}$" which simplifies to "$2 = 2$". So $P(1)$ is true.

Induction step: suppose $P(n)$ is true for some $n \geq 1$. That is, suppose $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$ for some $n \geq 1$. We want to prove $P(n+1)$ is true, that is prove $\sum_{i=1}^{n+1} i(i+1) = \frac{(n+1)(n+2)(n+3)}{3}$. Now

$$\sum_{i=1}^{n+1} i(i+1) = \sum_{i=1}^{n} i(i+1) + (n+1)(n+2) = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$$

$$= (n+1)(n+2)(\frac{n}{3} + 1) = \frac{(n+1)(n+2)(n+3)}{3}$$

This proves $P(n+1)$ is true. By the principle of induction, we conclude $P(n)$ is true for all $n \geq 1$.

4. **(6 pts)** Consider the following subsets of $\mathbb{Z}$:

$$A = \{x \in \mathbb{Z} \mid x^2 + 2x > 0\}, \quad B = \{x \in \mathbb{Z} \mid |x+1| > 1\}$$

Show that $A = B$.

Let $x \in A$. Then $x^2 + 2x > 0$. This implies $x(x+2) > 0$. This further implies either $x > 0$ and $x + 2 > 0$, or $x < 0$ and $x + 2 < 0$. The first case simplifies to $x > 0$. The second case simplifies to $x < -2$.

In the first case, $x > 0$ implies $x + 1 > 1$, which implies $|x + 1| > 1$. Hence $x \in B$. In the second case, $x < -2$ implies $x + 1 < -1$, which implies $|x + 1| > 1$. Hence $x \in B$.

In all cases, $x \in A$ implies $x \in B$, so $A \subseteq B$.

Let $x \in B$. Then $|x + 1| > 1$, which implies $x + 1 > 1$ or $x + 1 < -1$. The first case simplifies to $x > 0$. The second case simplifies to $x < -2$.

In the first case $x^2 + 2x > 0 + 2 \cdot 0 = 0$, so $x \in A$. In the second case, since $x < -2$, we have that x and $x + 2$ are negative, hence $x^2 + 2x = x(x + 2) > 0$, so $x \in A$.

In all cases, $x \in B$ implies $x \in A$, so $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, we have $A = B$.

5. **(6 pts)**

(a) **(1 pt)** Give the definition of an equivalence relation on a set X. If you use words like "transitive", briefly define them.

A relation $\sim$ on X is an equivalence relation if it is reflexive, symmetric and transitive.

The relation $\sim$ is reflexive if $\forall x \in X, x \sim x$.

The relation $\sim$ is symmetric if $\forall x, y \in X. x \sim y \to y \sim x$

The relation $\sim$ is transitive if $\forall x, y, z \in X. (x \sim y \wedge y \sim z) \to x \sim z$

(b) **(3 pts)** Show that the relation on $\mathbb{R}$ defined by

$$x \sim y \iff \exists q \in \mathbb{Z} \text{ such that } y = x + q$$

is an equivalence relation.

The relation is reflexive: let $x \in \mathbb{R}$. Then consider $q = 0$. In that case, $x = x + q$, so $x \sim x$.

The relation is symmetric: let $x, y \in \mathbb{R}$ such that $x \sim y$. There exists $q \in \mathbb{Z}$ such that $y = x + q$. This equation turns into $x = y - q$. Since $-q \in \mathbb{Z}$, we can conclude $y \sim x$.

The relation is transitive: let $x, y, z \in \mathbb{R}$ such that $x \sim y$ and $y \sim z$. There exists $q, q' \in \mathbb{Z}$ such that $y = x + q$ and $z = y + q'$. Combining the two equation gives $z = x + q + q'$. Since $q + q' \in \mathbb{Z}$, we can conclude $x \sim z$.

Since the relation is reflexive, symmetric and transitive, it is an equivalence relation.

(c) **(2 pts)** Show that for any $x \in \mathbb{R}$, there exists a real number $y \in [0, 1)$ such that $[x]_\sim = [y]_\sim$, where $\sim$ is the relation from (b).

Let $x \in \mathbb{R}$. Consider $\lfloor x \rfloor$, the greatest integer less than or equal to x. Take $y = x - \lfloor x \rfloor$. Since the difference between consecutive integers is 1, we conclude that $0 \leq y < 1$, hence $y \in [0, 1)$. Furthermore, take $q = -\lfloor x \rfloor$. So $q \in \mathbb{Z}$. Then $y = x + q$, so $x \sim y$. This shows $[x]_\sim = [y]_\sim$.

6. **(6 pts)**

(a) **(3 pts)** What are the solutions $x \in \mathbb{Z}$ of the equation

$$x^2 + x + 1 \equiv 2 \mod 5?$$

Hint: you can express the set of solutions in terms of the possible remainders of x upon division by 5.

We plug in all possible values of $x \mod 5$.

If $x \equiv 0 \mod 5$, then $x^2 + x + 1 \equiv 1 \not\equiv 2 \mod 5$. If $x \equiv 1 \mod 5$, then $x^2 + x + 1 \equiv 3 \not\equiv 2 \mod 5$. If $x \equiv 2 \mod 5$, then $x^2 + x + 1 \equiv 2 \mod 5$. If $x \equiv 3 \mod 5$, then $x^2 + x + 1 \equiv 3 \not\equiv 2 \mod 5$. If $x \equiv 4 \mod 5$, then $x^2 + x + 1 \equiv 1 \not\equiv 2 \mod 5$.

So the solutions are all the integers $x \in \mathbb{Z}$ such that $x \equiv 2 \mod 5$. That is $x = \ldots -8, -3, 2, 7, \ldots$.

(b) **(3 pts)** What is the last digit of 3^{33}? That is, calculate the remainder upon division by 10.

We notice that $3^4 = 81 \equiv 1 \mod 10$. So $3^{33} = 3^{32} \cdot 3 = (3^4)^8 \cdot 3 \equiv 1^8 \cdot 3 \equiv 3 \mod 10$.

7. **(5 pts)** Consider the subset S of $\mathbb{R}$ given as follows:

$$S := \left\{ 2 + \frac{1}{3x} \mid x \in \mathbb{R} \text{ and } x \geq 2 \right\}.$$

(a) **(2 pts)** Find the maximum of S, with justification, or prove that it does not exist.
We have that $\max(S) = \frac{13}{6}$. Indeed, $\frac{13}{6} \in S$ (take $x = 2$). Further more, we have

$$x \geq 2 \rightarrow 3x \geq 6 \rightarrow \frac{1}{3x} \leq \frac{1}{6} \rightarrow 2 + \frac{1}{3x} \leq 2 + \frac{1}{6} = \frac{13}{6}$$

hence every element of S is less than or equal to $\frac{13}{6}$. This proves $\max(S) = \frac{13}{6}$.

(b) **(1 pt)** Find the supremum of S, with justification, or prove that it does not exist.
Since the maximum of S exists, so does the supremum of S, and $\sup(S) = \max(S) = \frac{13}{6}$.

(c) **(2 pts)** Find the infimum of S, with justification, or prove that it does not exist.
We show that $\inf(A) = 2$. We first show 2 is a lower bound of S. We have

$$x \geq 2 \to x > 0 \to 3x > 0 \to \frac{1}{3x} > 0 \to 2 + \frac{1}{3x} > 2$$

hence every element of S is greater than 2. So 2 is a lower bound for S. We now show 2 is the greatest lower bound for S. Suppose $b > 2$ is a lower bound for S. We have

$$x \geq 2 \to 2 + \frac{1}{3x} > b \to \frac{1}{3x} > b - 2 \to 3x < \frac{1}{b-2}$$

the last implication is possible because $b - 2$ is positive. Then we get $x < \frac{1}{3(b-2)}$, for all $x \geq 2$. So $\frac{1}{3(b-2)}$ is a real number greater than any $x \geq 2$. This is a contradiction, hence b is not a lower bound of S. We must conclude $\inf(A) = 2$.

8. **(6 pts)** Consider the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{2x} - 1.$$

Justify your answers to the following questions.

(a) **(2 pts)** Is f injective?

Yes. Suppose that $x, y \in \mathbb{R}_{>0}$ are such that $f(x) = f(y)$. Then

$$\begin{aligned} f(x) = f(y) &\Rightarrow \frac{1}{2x} - 1 = \frac{1}{2y} - 1 \\ &\Rightarrow \frac{1}{2x} = \frac{1}{2y} && \text{(add 1 to both sides)} \\ &\Rightarrow 2x = 2y && \text{(invert both sides)} \\ &\Rightarrow x = y && \text{(cancellation)} \end{aligned}$$

(b) **(2 pts)** What is the image of f? Use interval notation in your final answer.

The image is $f(\mathbb{R}_{>0}) = \{\frac{1}{2x} - 1 \mid x \in \mathbb{R}_{>0}\}$. Note that

$$\begin{aligned} x > 0 &\iff 2x > 0 \\ &\iff \frac{1}{2x} > 0 \\ &\iff \frac{1}{2x} - 1 > -1 \end{aligned}$$

Thus $x > 0$ if and only if $f(x) > -1$, i.e., $f(\mathbb{R}_{>0}) = (-1, \infty)$. No f is not surjective, since $f(\mathbb{R}_{>0}) \neq \mathbb{R}$.

(c) **(1 pt)** Is f bijective?

No, f is not surjective, therefore it is not bijective.

(d) **(1 pt)** Does f have a left or right inverse?

Since f is injective, it has a left inverse. Since f is not surjective, it does not have a right inverse.

9. **(7 pts)**

(a) **(1 pt)** State what it means for a sequence $(x_n)_{n=1}^{\infty}$ of real numbers to converge to $L \in \mathbb{R}$.

The sequence $(x_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}. \forall n \geq N, |x_n - L| < \varepsilon).$$

(b) **(3 pts)** Prove, using the definition of a limit, that

$$\lim_{n\to\infty} \left(13 + \frac{2}{4+n^2}\right) = 13.$$

Let $\epsilon > 0$. Take $N > \frac{2}{\epsilon}$. Such a natural number N exists since $\mathbb{N}$ is not bounded above. Then, for all $n \geq N$:

$$\begin{aligned}
\left|13 + \frac{2}{4+n^2} - 13\right| &= \left|\frac{2}{4+n^2}\right| \\
&= \frac{2}{4+n^2} && \text{(Numerator and denominator are positive)} \\
&\leq \frac{2}{n^2} && \text{(since } 4+n^2 \geq n^2\text{)} \\
&\leq \frac{2}{n} && \text{(since } n \geq 1\text{)} \\
&\leq \frac{2}{N} && \text{(since } n \geq N\text{)} \\
&< \frac{2}{\frac{2}{\epsilon}} && \text{(since } N > \frac{\epsilon}{2}\text{)} \\
&= \epsilon
\end{aligned}$$

This N fulfills the definition of a limit, therefore the limit is 13.

(c) **(3 pts)** Consider the sequence $(x_n)_{n=1}^{\infty}$ defined recursively by $x_1 = 3$ and, for $n \geq 1$,

$$x_{n+1} = \begin{cases} x_n + 1 & \text{if } n \text{ is even} \\ x_n - 1 & \text{if } n \text{ is odd} \end{cases}$$

Does the sequence $(x_n)_{n=1}^{\infty}$ converge or diverge? Justify your answer.

The sequence diverges. Using a proof by contradiction, assume $\lim_{k\to\infty} x_k = L$ for some $L \in \mathbb{R}$. Then, for $\epsilon = \frac{1}{3}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$. Consider $n \geq N$ such that n is even. Then

$$1 = |x_n + 1 - x_n| = |x_{n+1} - x_n| \leq |x_{n+1} - L| + |x_n - L| \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

We thus have $1 < \frac{2}{3}$, which is a contradiction. We must conclude that the sequence diverges.