Q1. a) [1 POINT] Let y represent an integer. Use the Binomial Theorem to express the expansion of the following binomial as a finite series. Do not expand the expression; simply write it as a finite series, using appropriate summation notation. No justification is required for this part.

$$(y-1)^{42} = \ldots$$

b)[2 POINTS] Evaluate the following finite series. Calculators are NOT permitted, nor are they needed.

$$\sum_{j\in\{1,3,5\}} \binom{j+4}{5-j}$$ Show ALL your steps! The final answer should be a single integer.

c) [2 POINTS] Let $I = \Big\{S : S \subseteq \{1,3,5\} \text{ and } |S| = 2\Big\}$. Using this index set, fully evaluate the following finite series:

$$\sum_{S\in I}\left(\prod_{a_i\in S} a_i\right)$$ Show ALL your steps! The final answer should be a single integer.

Q2. a) [1 POINT] State the **Well-Ordering Principle.** Be precise!

b) [2 POINTS] Let $S = \{k \in \mathbb{N} : \exists\, x, y \in \mathbb{Z} \text{ such that } k = 30x + 12y\}$.

Briefly and clearly justify **why** S has a **smallest element**. (you do NOT need to find $\min(S)$).

Q3. [4 POINTS] Consider the following three sets:

$$W = \{20k + 7 : k \in \mathbb{Z}\}, \quad A = \{10m + 7 : m \in \mathbb{Z}\} \quad S = \{5n + 2 : n \in \mathbb{Z}\}.$$

Rigorously prove that $(W \times A) \subseteq (A \times S)$.

Be sure to use appropriate mathematical notation throughout and briefly justify each step of your proof.

Q4. Let $\sim$ be a relation on $\mathbb{Z}$, defined as follows:

$$\forall a, b \in \mathbb{Z}, \qquad a \sim b \quad \Longleftrightarrow \quad 5 \mid (a + 4b)$$

a) [5 POINTS] Carefully prove that $\sim$ is an **equivalence relation** on $\mathbb{Z}$.

b) [2 POINTS] Find 2 distinct elements that belong to the **equivalence class** $[-1]_\sim$.

Briefly justify each of your answers.

Q5.(a) **[2 POINTS]** Use modular arithmetic to evaluate the following expression in $\mathbb{Z}_6$.

Give your answer in terms of one of the canonical representatives $[0], \ldots, [5]$ and show your steps:

$$\big([-2] \oplus [41]\big) \odot \big([14] \odot [17]\big)$$

(b) **[2 POINTS]** Does $[4] \in \mathbb{Z}_9$ have a **multiplicative inverse** in $\mathbb{Z}_9$?

If so, give $[4]^{-1}$ in terms of its canonical representative $[0], [1], \ldots, [8]$ and briefly justify your answer. If not, briefly justify.

(c) **[2 POINTS]** Does $[6] \in \mathbb{Z}_9$ have an **additive inverse** in $\mathbb{Z}_9$?

If so, give $-[6]$ in terms of its canonical representative $[0], [1], \ldots, [8]$ and briefly justify your answer.. If not, briefly justify.

Q6. **[5 POINTS]** Prove the following statement:

If $c, a, y \in \mathbb{R}$ and $y + 1 \neq 0$, then $\dfrac{(c+a)y}{y+1} + \dfrac{a+c}{1+y} = c + a.$

Do NOT use any propositions stated in class or DGDs.

Each step should be clearly justified with a single axiom of $\mathbb{R}$ or the definition of division.

Be specific when you justify each step: <u>name</u> the axiom or definition that is being applied.
