**Q1.** a) **[1 POINT]** Let $y$ represent an integer. Use the Binomial Theorem to express the expansion of the following binomial as a finite series. Do not expand the expression; simply write it as a finite series, using appropriate summation notation. No justification is required for this part.

$$(y-1)^{42} = \dots$$

Solution:

$$(y-1)^{42} = \sum_{m=0}^{42} \binom{42}{m} y^m (-1)^{42-m}$$

b) **[2 POINTS]** Evaluate the following finite series. Calculators are NOT permitted, nor are they needed.

$$\sum_{j \in \{1,3,5\}} \binom{j+4}{5-j}$$

Show ALL your steps! The final answer should be a single integer.

Solution: We have

$$\begin{aligned}
\sum_{j \in \{1,3,5\}} \binom{j+4}{5-j} &= \binom{1+4}{5-1} + \binom{3+4}{5-3} + \binom{5+4}{5-5} \\
&= \binom{5}{4} + \binom{7}{2} + \binom{9}{0} \\
&= \frac{5!}{4!1!} + \frac{7!}{2!5!} + \frac{9!}{0!9!} \\
&= \frac{5 \cdot 4!}{4!1!} + \frac{7 \cdot 6 \cdot 5!}{2!5!} + \frac{1}{0!} \\
&= \frac{5}{1} + \frac{7 \cdot 6}{2 \cdot 1} + \frac{1}{1} \\
&= 5 + 21 + 1 \\
&= 27
\end{aligned}$$

c) **[2 POINTS]** Let $I = \Big\{ S : S \subseteq \{1,3,5\} \text{ and } |S| = 2 \Big\}$. Using this index set, fully evaluate the following finite series:

$$\sum_{S \in I} \left( \prod_{a_i \in S} a_i \right)$$

Show ALL your steps! The final answer should be a single integer.

Solution: First, $I = \Big\{ S : S \subseteq \{1,3,5\} \text{ and } |S| = 2 \Big\} = \Big\{ \{1,3\}, \{1,5\}, \{3,5\} \Big\}$, so we have:

$$\begin{aligned}
\sum_{S \in I} \left( \prod_{a_i \in S} a_i \right) &= \prod_{a_i \in \{1,3\}} a_i + \prod_{a_i \in \{1,5\}} a_i + \prod_{a_i \in \{3,5\}} a_i \\
&= (1)(3) + (1)(5) + (3)(5) \\
&= 23
\end{aligned}$$

**Q2.** a) **[1 POINT]** State the **Well-Ordering Principle.** Be precise!

Solution: Every nonempty subset of $\mathbb{N}$ has a smallest element.

b) **[2 POINTS]** Let $S = \{k \in \mathbb{N} : \exists\, x, y \in \mathbb{Z} \text{ such that } k = 30x + 12y\}$.

Briefly and clearly justify **why** $S$ has a **smallest element**. (you do NOT need to find $\min(S)$).

Solution: By its definition, $S$ is a subset of $\mathbb{N}$.
$S$ is nonempty since, for example, $30(1) + 12(1) \in S$.
By the Well-ordering Principle, $S$ must contain a smallest element.

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**Q3.** **[4 POINTS]** Consider the following three sets:

$$W = \{20k + 7 : k \in \mathbb{Z}\}, \quad A = \{10m + 7 : m \in \mathbb{Z}\} \quad S = \{5n + 2 : n \in \mathbb{Z}\}.$$

Rigorously prove that $(W \times A) \subseteq (A \times S)$.

Be sure to use appropriate mathematical notation throughout and briefly justify each step of your proof.

Solution:

Assume $(a, b) \in W \times A$.

Then $a \in W$ and $b \in A$ by def. of $\times$

$\implies a = 20k + 7$ and $b = 10m + 7$ for some $k, m \in \mathbb{Z}$, by def. of $W$ and $A$

$\implies a = 10(2k) + 7$ and $b = 5(2m + 1) + 2$ where $2k, 2m + 1 \in \mathbb{Z}$ since $1, 2, k, m \in \mathbb{Z}$.

$\implies a \in A$ and $b \in S$ by def. of $A$ and $S$, respectively

$\implies (a, b) \in A \times S$ by def. of $\times$.

This proves $(a, b) \in W \times A \implies (a, b) \in A \times S$. Therefore, $W \times A \subseteq A \times S$. $\square$

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**Q4.** Let $\sim$ be a relation on $\mathbb{Z}$, defined as follows:

$$\forall a, b \in \mathbb{Z}, \qquad a \sim b \quad \iff \quad 5 \mid (a + 4b)$$

a) **[5 POINTS]** Carefully prove that $\sim$ is an **equivalence relation** on $\mathbb{Z}$.

Solution: We must prove that $\sim$ is reflexive, symmetric and transitive.

[reflexivity] Let $a \in \mathbb{Z}$. Then $a + 4a = 5a$ which is divisible by $5$. Therefore, $a \sim a$.

Since $\forall a \in \mathbb{Z}$, $a \sim a$, $\sim$ is reflexive.

[symmetry] Let $a, b \in \mathbb{Z}$. Assume $a \sim b$.

Then $5 \mid (a + 4b)$ — by def. of $\sim$

$\implies \exists k \in \mathbb{Z}$ such that $a + 4b = 5k$ — by def. of divides

$$\begin{aligned} \implies b + 4a &= b + 4(5k - 4b) \\ &= -15b + 5(4k) \\ &= 5(-3b + 4k) \end{aligned}$$

$\implies 5 \mid (b + 4a)$ — since $-3b + 4k \in \mathbb{Z}$.

$\implies b \sim a$ — Thus, $\sim$ is symmetric.

[transitivity] Let $a, b, c \in \mathbb{Z}$. Assume $a \sim b$ and $b \sim c$.

| | |
|---|---|
| Then $5 \mid (a+4b)$ and $5 \mid (b+4c)$ | by def. of $\sim$ |
| $\implies \exists k, l \in \mathbb{Z}$ such that $a + 4b = 5k$ and $b + 4c = 5l$ | by def. of divides |
| $\implies (a+4b) + (b+4c) = 5k + 5l$ | |
| $\implies a + 5b + 4c = 5k + 5l$ | |
| $\implies a + 4c = 5(k + l - b)$ | |
| $\implies 5 \mid (a+4c)$ | since $k + l - b \in \mathbb{Z}$ |
| $\implies a \sim c$ | Thus, $\sim$ is transitive. |

b) **[2 Points]** Find 2 distinct elements that belong to the **equivalence class** $[-1]_\sim$.

Briefly justify each of your answers.

Solution:

$$\begin{aligned}[-1]_\sim &= \{x \in \mathbb{Z} : x \sim -1\}\\ &= \{x \in \mathbb{Z} :\ 5 \mid (x + 4(-1))\}\\ &= \{x \in \mathbb{Z} : 5 \mid (x-4)\}\\ &= \{x \in \mathbb{Z} : x - 4 = 5k \text{ for some } k \in \mathbb{Z}\}\\ &= \{x \in \mathbb{Z} : x = 5k + 4 \text{ for some } k \in \mathbb{Z}\}\end{aligned}$$

Thus, all integers $x$ of the form $x = 5k + 4$, for some $k \in \mathbb{Z}$ are elements of $[-1]_\sim$

For example, $-1 \in [-1]_\sim$, $\quad 4 \in [-1]_\sim$ $\quad$ (there are many other possible answers)

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**Q5.**(a) **[2 points]** Use modular arithmetic to evaluate the following expression in $\mathbb{Z}_6$.

Give your answer in terms of one of the canonical representatives $[0], \ldots, [5]$ and show your steps:

$$\big([-2] \oplus [41]\big) \odot \big([14] \odot [17]\big)$$

Solution: In $\mathbb{Z}_6$, we have

$$\begin{aligned}\big([41] \oplus [-2]\big) \odot \big([17] \odot [14]\big) &= \big([5] \oplus [4]\big) \odot \big([5] \oplus [2]\big)\\ &= [5+4] \odot [(5)(2)]\\ &= [9] \odot [10]\\ &= [3] \odot [4]\\ &= [(3)(4)]\\ &= [12]\\ &= [0]\end{aligned}$$

(b) **[2 points]** Does $[4] \in \mathbb{Z}_9$ have a **multiplicative inverse** in $\mathbb{Z}_9$?

If so, give $[4]^{-1}$ in terms of its canonical representative $[0], [1], \ldots, [8]$ and briefly justify your answer. If not, briefly justify.

Solution: Yes, $[4]$ has a multiplicative inverse in $\mathbb{Z}_9$ since $[4] \odot [7] = [28] = [1]$. Thus, $[7]$ is a multiplicative inverse of $[4]$ in $\mathbb{Z}_9$.

(c) **[2 POINTS]** Does $[6] \in \mathbb{Z}_9$ have an **additive inverse** in $\mathbb{Z}_9$?

If so, give $-[6]$ in terms of its canonical representative $[0], [1], \ldots, [8]$ and briefly justify your answer.. If not, briefly justify.

Solution: Yes, $[4]$ has an additive inverse in $\mathbb{Z}_9$ since $[6] \oplus [3] = [9] = [0]$ . Thus, $[3]$ is an additive inverse of $[6]$ in $\mathbb{Z}_9$.

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**Q6. [5 POINTS]** Prove the following statement:

$$\text{If } c, a, y \in \mathbb{R} \text{ and } y + 1 \neq 0, \quad \text{then} \quad \frac{(c+a)y}{y+1} + \frac{a+c}{1+y} = c + a.$$

Do NOT use any propositions stated in class or DGDs.

Each step should be clearly justified with a single axiom of $\mathbb{R}$ or the definition of division.

Be specific when you justify each step: <u>name</u> the axiom or definition that is being applied.

Solution: Assume $c, a, y \in \mathbb{R}$ and $y + 1 \neq 0$.

Then $(y+1)^{-1}$ exists by the Multiplicative Inverse Axiom, and we have:

$$\begin{aligned}
\frac{(c+a)y}{y+1} + \frac{a+c}{y+1} &= ((c+a)y)(y+1)^{-1} + (a+c)(y+1)^{-1} && \text{by def. of division (twice)} \\
&= (y+1)^{-1}((c+a)y) + (y+1)^{-1}(a+c) && \text{Commut. of Mult. (twice)} \\
&= (y+1)^{-1}\Big((c+a)y + (a+c)\Big) && \text{Distributivity} \\
&= (y+1)^{-1}\Big((c+a)y + (a+c)\cdot 1\Big) && \text{Mult. Identity Ax.} \\
&= (y+1)^{-1}\Big((c+a)y + (c+a)\cdot 1\Big) && \text{Commut. of Addition} \\
&= (y+1)^{-1}\Big((c+a)(y+1)\Big) && \text{Distrib.} \\
&= \Big((c+a)(y+1)\Big)(y+1)^{-1} && \text{Commut. of Mult.} \\
&= (c+a)\Big((y+1)(y+1)^{-1}\Big) && \text{Assoc. of Mult.} \\
&= (c+a)\cdot 1 && \text{Mult. Inverse Ax.} \\
&= c + a && \text{Mult. Identity Ax.} \quad \square
\end{aligned}$$

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