Q1. [6 POINTS] Let P be the following implication, where $a, b, c \in \mathbb{Z}$.

$$P: \quad \text{If} \quad a \mid bc, \quad \text{then} \quad a \mid b \text{ or } a \mid c.$$

a) Provide a counterexample (with concrete values of $a, b, c \in \mathbb{Z}$) to show that P is not true in general. Briefly explain how your counterexample shows that P can be false.

b) Write the **converse** of P, and determine if it is true or false. If the converse of P is true, prove it directly. Otherwise, provide a counterexample and briefly explain.

c) Write the **contrapositive** of P. Determine whether it is true for all $a, b, c \in \mathbb{Z}$. For this part only, you do not need to justify your answer.

Q2. [3 POINTS] Write the **negation** of each of the following statements. Do not simply write "not" or $\neg$ in front of the statement. Simplify each statement's negation appropriately.

a) $(\forall m \in \mathbb{Z})(\exists n \in \mathbb{Z} \text{ such that })\, m + n = 0.$

b) $(\exists x \in \mathbb{Z} \text{ such that })\Big[x \neq 0 \text{ and } \big(\forall y \in \mathbb{Z},\ xy = y\big)\Big]$

c) If all premises are true, then the conclusion is true.

Q3. Prove each of the following statements.

Do not use any propositions that we proved in class. Each step should be clearly justified with a single axiom of $\mathbb{Z}$, the definition of subtraction, a property of $=$, or a logical implication of a previous step. Be specific when you justify each step.

a) [3 POINTS] If $\quad w, x, y \in \mathbb{Z}, \quad$ then $\quad (wx)(y + 1) = x(w + yw).$

b) [5 POINTS] Let $a, w, d \in \mathbb{Z}$.

$$\text{If} \quad a(wd) = wa - dw \text{ and } w \neq 0, \quad \text{then} \quad d(a + 1) = a.$$

Q4. [4 POINTS] Let $K, M, L \in \mathbb{Z}$. Prove the following statement:

$$\text{If} \quad K < KM + L \text{ and } L \leq 0, \quad \text{then} \quad LKM \leq LK.$$

Do not use any propositions we proved about inequalities. Use ONLY the definitions of $<$ and $\leq$, the axioms of the natural numbers and the arithmetic of $\mathbb{Z}$ (Ch. 1 material). You do not need to show all steps related to the arithmetic of $\mathbb{Z}$.

Q5. **[5 POINTS]** Use a proof by (weak) induction to show the following holds:

$$\forall n \in \mathbb{N}, \quad \sum_{j=1}^{n}(6j^2 - 2j) = 2n^2(n+1).$$

Clearly indicate what variables represent throughout your proof. Clearly state your induction hypothesis. Clearly indicate when you use the induction hypothesis.

Q6. **[4 POINTS]** For each integer $n \geq 12$, let $P(n)$ be the statement below:

$$P(n): \quad (\exists a, b \in \mathbb{Z}_{\geq 0} \text{ such that }) \; n = 4a + 5b.$$

Using a proof by **strong induction**, show that $P(n)$ is true for all integers $n \geq 12$.

Clearly indicate what variables represent throughout your proof. Clearly state your strong induction hypothesis. Clearly indicate when you use the induction hypothesis.