Q1. **[6 POINTS]** Let P be the following implication, where $a, b, c \in \mathbb{Z}$.

$$P: \quad \text{If} \quad a \mid bc, \quad \text{then} \quad a \mid b \text{ or } a \mid c.$$

a) Provide a counterexample (with concrete values of $a, b, c \in \mathbb{Z}$) to show that P is not true in general. Briefly explain how your counterexample shows that P can be false.

Solution: There are many possible counterexamples. For instance, take $a = 6, b = 2, c = 3$. Then $bc = 6$. We have $a \mid ab$ since $6 \mid 6$ is true.

However, $a \nmid b$ since $6 \nmid 2$ and $a \nmid c$ since $6 \nmid 3$. Therefore, the or-statement "$a \mid b$ or $a \mid c$" is false.

For these values of a, b, c, the premise of P is true but its conclusion is false.

b) Write the **converse** of P, and determine if it is true or false. If the converse of P is true, prove it directly. Otherwise, provide a counterexample and briefly explain.

Solution: Converse of P: If $a \mid b$ or $a \mid c$, then $a \mid bc$.

The converse is true. Proof: Let $a, b, c \in \mathbb{Z}$. Assume $a \mid b$ or $a \mid c$.

Case 1. If $a \mid b$, then $\exists q \in \mathbb{Z}$ such that $b = qa$. Consequently, $bc = (qa)c = (qc)a$. Since $q, c \in \mathbb{Z}$, this shows that $a \mid bc$ in this case.

Case 2. If $a \mid c$, then $\exists q \in \mathbb{Z}$ such that $c = qa$. Consequently, $bc = b(qa) = (qb)a$. Since $q, b \in \mathbb{Z}$, this shows that $a \mid bc$ in this case.

In both cases, $a \mid bc$. □

c) Write the **contrapositive** of P. Determine whether it is true for all $a, b, c \in \mathbb{Z}$. For this part only, you do not need to justify your answer.

Solution: Contrapositive of P: If $a \nmid b$ and $a \nmid c$, then $a \nmid bc$.

Since the implication P is logically equivalent to its contrapositive, the contrapositive of P is also not true for all $a, b, c \in \mathbb{Z}$.

Q2. **[3 POINTS]** Write the **negation** of each of the following statements. Do not simply write "not" or $\neg$ in front of the statement. Simplify each statement's negation appropriately.

a) $(\forall m \in \mathbb{Z})(\exists n \in \mathbb{Z} \text{ such that }) \, m + n = 0.$

Solution: Negation:$(\exists m \in \mathbb{Z} \text{ such that })(\forall n \in \mathbb{Z}) \, m + n \neq 0.$

b) $(\exists x \in \mathbb{Z} \text{ such that }) \Big[x \neq 0 \text{ and } \big(\forall y \in \mathbb{Z}, \; xy = y\big) \Big]$

Solution: Negation: $(\forall x \in \mathbb{Z}) \big[x = 0 \text{ or } (\exists y \in \mathbb{Z} \text{ such that }) xy \neq y \big]$

c) If all premises are true, then the conclusion is true.

Solution: Negation: All premises are true and the conclusion is false.

Q3. Prove each of the following statements.

Do not use any propositions that we proved in class. Each step should be clearly justified with a single axiom of $\mathbb{Z}$, the definition of subtraction, a property of $=$, or a logical implication of a previous step. Be specific when you justify each step.

a) **[3 Points]** If $w, x, y \in \mathbb{Z}$, then $(wx)(y+1) = x(w+yw)$.

Solution: Assume $w, x, y \in \mathbb{Z}$. Then

$$
\begin{aligned}
(wx)(y+1) &= (xw)(y+1) && \text{Commut. of Mult.}\\
&= x\big(w(y+1)\big) && \text{Assoc. of Mult.}\\
&= x\big(wy + w \cdot 1\big) && \text{Distrib.}\\
&= x\big(wy + w\big) && \text{Mult. Identity Ax.}\\
&= x\big(w + wy\big) && \text{Commut. of Add.}\\
&= x\big(w + yw\big) && \text{Commut. of Mult.}
\end{aligned}
$$

b) **[5 Points]** Let $a, w, d \in \mathbb{Z}$.

If $a(wd) = wa - dw$ and $w \neq 0$, then $d(a+1) = a$.

Solution: Let $a, w, d \in \mathbb{Z}$. Assume $a(wd) = wa - dw$ and $w \neq 0$. Then:

$$
\begin{aligned}
& a(wd) = wa - dw && \text{by assumption}\\
\implies\ & a(wd) = wa + (-dw) && \text{by def. of Subtraction}\\
\implies\ & a(wd) + dw = \big(wa + (-dw)\big) + dw && \text{replacement}\\
\implies\ & a(wd) + dw = wa + \big((-dw) + dw\big) && \text{Assoc. of Add.}\\
\implies\ & a(wd) + dw = wa + \big(dw + (-dw)\big) && \text{Commut. of Add.}\\
\implies\ & a(wd) + dw = wa + 0 && \text{Add. Inverse Ax.}\\
\implies\ & a(wd) + dw = wa && \text{Add. Identity Ax.}\\
\implies\ & (wd)a + dw = wa && \text{Commut. of Mult.}\\
\implies\ & w(da) + dw = wa && \text{Assoc. of Mult.}\\
\implies\ & w(da) + wd = wa && \text{Commut. of Mult.}\\
\implies\ & w\big(da + d\big) = wa && \text{Distrib.}\\
\implies\ & da + d = a && \text{Cancellation (since } w \neq 0\text{)}\\
\implies\ & da + d \cdot 1 = a && \text{Mult. Identity Ax.}\\
\implies\ & d(a+1) = a && \text{Distrib.} \quad \square
\end{aligned}
$$

Q4. **[4 Points]** Let $K, M, L \in \mathbb{Z}$. Prove the following statement:

If $K < KM + L$ and $L \leq 0$, then $LKM \leq LK$.

Do not use any propositions we proved about inequalities. Use ONLY the definitions of $<$ and $\leq$, the axioms of the natural numbers and the arithmetic of $\mathbb{Z}$ (Ch. 1 material). You do not need to show all steps related to the arithmetic of $\mathbb{Z}$.

Solution: Let $K, M, L \in \mathbb{Z}$. Assume $K < KM + L$ and $L \leq 0$.

Then $KM + L - K \in \mathbb{N}$ by definition of $<$, and $L = 0$ or $L < 0$ by definition of $\leq$.

Case 1: Assume $L = 0$. Then $LKM = 0KM = 0$ and $LK = 0K = 0$. Thus, $LKM = LK$, hence $LKM \leq LK$ in Case 1.

Case 2: Assume $L < 0$. Then $0 - L = -L \in \mathbb{N}$ by definition of $<$.

Since $\mathbb{N}$ is closed under addition, it follows that $(KM + L - K) + (-L) \in \mathbb{N}$, thus $KM - K \in \mathbb{N}$.

Since $\mathbb{N}$ is closed under multiplication, it follows that $(-L)(KM - K) \in \mathbb{N}$. Thus, $LK - LKM \in \mathbb{N}$, which shows that $LKM < LK$ in this case. Therefore, $LKM \leq LK$ in Case 2. □

Q5. **[5 POINTS]** Use a proof by (weak) induction to show the following holds:

$$\forall n \in \mathbb{N}, \quad \sum_{j=1}^{n} (6j^2 - 2j) = 2n^2(n+1).$$

Clearly indicate what variables represent throughout your proof. Clearly state your induction hypothesis. Clearly indicate when you use the induction hypothesis.

Solution: For each $n \in \mathbb{N}$, let $P(n)$ be the statement $\sum_{j=1}^{n} (6j^2 - 2j) = 2n^2(n+1)$.

Base case: For $n = 1$, we have $\sum_{j=1}^{n} (6j^2 - 2j) = \sum_{j=1}^{1} (6j^2 - 2j) = 6(1^2) - 2(1) = 4$, and $2n^2(n+1) = 2(1^2)(1+1) = 4$.

Therefore, $\sum_{j=1}^{n} (6j^2 - 2j) = 2n^2(n+1)$ for $n = 1$, i.e. $P(1)$ holds.

IH: Let $n \in \mathbb{N}$ be an arbitrary natural number. Assume $\sum_{j=1}^{n}(6j^2 - 2j) = 2n^2(n+1)$ holds. Then

$$\begin{aligned}
\sum_{j=1}^{n+1}(6j^2 - 2j) &= \sum_{j=1}^{n}(6j^2 - 2j) + \sum_{j=n+1}^{n+1}(6j^2 - 2j) \\
&= \left[\sum_{j=1}^{n}(6j^2 - 2j)\right] + \left[6(n+1)^2 - 2(n+1)\right] \\
&= \left[2n^2(n+1)\right] + \left[6(n+1)^2 - 2(n+1)\right] && \text{by IH!} \\
&= (n+1)\left[2n^2 + 6(n+1) - 2\right] \\
&= (n+1)(2)(n^2 + 3(n+1) - 1) \\
&= 2(n+1)(n^2 + 3n + 2) \\
&= 2(n+1)(n+1)(n+2) \\
&= 2(n+1)^2((n+1)+1)
\end{aligned}$$

Therefore, $\sum_{j=1}^{n+1}(6j^2 - 2j) = 2(n+1)^2((n+1)+1)$, hence $P(n+1)$ holds.

This complete the proof of the induction step. $\square$

Q6. **[4 POINTS]** For each integer $n \geq 12$, let $P(n)$ be the statement below:

$$P(n): \quad (\exists a, b \in \mathbb{Z}_{\geq 0} \text{ such that })\ n = 4a + 5b.$$

Using a proof by **strong induction**, show that $P(n)$ is true for all integers $n \geq 12$.

Clearly indicate what variables represent throughout your proof. Clearly state your strong induction hypothesis. Clearly indicate when you use the induction hypothesis.

Solution: Base cases:

(n=12) Since $12 = 4(3) + 5(0)$ and $3, 0 \in \mathbb{Z}_{\geq 0}$, $P(12)$ is true.

(n=13) Since $13 = 4(2) + 5(1)$ and $2, 1 \in \mathbb{Z}_{\geq 0}$, $P(13)$ is true.

(n=14) Since $14 = 4(1) + 5(2)$ and $1, 2 \in \mathbb{Z}_{\geq 0}$, $P(14)$ is true.

(n=15) Since $15 = 4(0) + 5(3)$ and $0, 3 \in \mathbb{Z}_{\geq 0}$, $P(15)$ is true.

IH: Let k be an arbitrary integer such that $k \geq 15$. Assume $\bigwedge_{j=12}^{k} P(j)$ is true.

Then $k + 1 - 4 \geq 12$ so $P(k+1-4)$ holds by IH.

Thus, $\exists a, b \in \mathbb{Z}$ such that $k + 1 - 4 = 4a + 5b$.

Since $k + 1 = k + 1 - 4 + 4$, we see that $k + 1 = 4a + 5b + 4 = 4(a+1) + 5b$. Since $a \in \mathbb{Z}_{\geq 0}$, it follows that $a + 1 \in \mathbb{Z}_{\geq 0}$. This shows that $P(k+1)$ holds, which completes the proof of the induction step. $\square$