Q1. [**5 points**] Carefully prove the following proposition using only the axioms of $\mathbb{Z}$, the properties of $=$, and the definition of subtraction.

Proposition. Let $r, s, t, u \in \mathbb{Z}$. If $r \neq 0$ and $r\big(s - (tu)\big) = t(rs)$, then $s = t(s + u)$.

Clearly indicate your assumptions and make sure to justify each step of your proof by naming one axiom of $\mathbb{Z}$ or by citing "replacement" or "definition of subtraction".
Do NOT use any propositions we proved in class.

Q2. [**4 points**] Write the negation of each of the following statements. Do not simply write "not" or $\neg$ in front of the statement, i.e. simplify each statement's negation appropriately.

(a) $(\exists\, b \in \mathbb{Q}$ such that $)(\forall m, n \in \mathbb{Z})\ \left[\frac{m}{n} = b \text{ and } m < n\right]$.

(b) There exist real numbers x, y such that $x^2 > y$ or $y + 1 = x$.

(c) If Moon is barking, then Moon is getting attention.

(d) $x = 0 \iff (\exists y \in \mathbb{N}$ such that $)\ y > x$

Q3. [**5 points**] Let $a, b \in \mathbb{Z}$ and consider the following implication P:

P: If $5 \mid a$ and $12 \mid (2b + 6)$, then $15 \mid ab$.

(a) Is P true for all $a, b \in \mathbb{Z}$? Fully justify your answer with a proof or counterexample and explanation.

(b) State the **contrapositive** of P. Is the contrapositive of P true for all $a, b \in \mathbb{Z}$? Fully justify your answer.

(c) State the **converse** of P. Is the converse of P true for all $a, b \in \mathbb{Z}$? Fully justify your answer.

Q4. [**5 points**] Consider the infinite integer sequence $(x_n)_{n=0}^{\infty}$ defined recursively, as follows:

$$x_0 = 1$$
$$\text{for all } n \in \mathbb{N}, \quad x_n = x_{n-1} + (2n + 1)^2$$

Use a **proof by induction** to show that $x_n = \dfrac{(n+1)(2n+1)(2n+3)}{3}$ for all $n \in \mathbb{N}$.

Your proof must be well-organized and each step must be appropriately justified. Clearly state your induction hypothesis and indicate "by IH" when it is used in your induction step.

Q5. [**4 points**] Determine whether each of the following statements is true for arbitrary sets A, B. If the statement is true, give an **indirect proof**. If the statement is false, give a **counterexample and explanation**. For your counterexample, give concrete sets A and B, (subsets of $\{1, 2, 3\}$ will suffice) and clearly explain how they show the statement can be false.

(a) If $A \times B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.

(b) If $A - B = \emptyset$, then $B \subseteq A$.

Q6. [3 points] Let S, T, W be arbitrary sets.

Give a rigorous proof to show that $(T \cup W) - S = (T - S) \cup (W - S)$.

Clearly justify each step! Be sure to use appropriate mathematical notation throughout.

Q7. [6 points] Let $\sim$ be a relation on $\mathbb{Z} \times \mathbb{Z}$ defined by the following rule:

$$\forall (a,b),(c,d) \in \mathbb{Z} \times \mathbb{Z}, \quad (a,b) \sim (c,d) \iff a < c \text{ and } b < d.$$

For this question, you must use the formal definition of $<$ and the axiom related to $\mathbb{N}$ to justify your answers. You may also use the following proposition, seen in class:

Prop. If $a \in \mathbb{Z}$, then exactly one (one and only one) of the following statements is true:

$$a \in \mathbb{N} \quad \text{or} \quad a = 0 \quad \text{or} \quad -a \in \mathbb{N}.$$

You do not need to show all steps related to the first five axioms of $\mathbb{Z}$. For example, you can simplify $(a - b) + b = a$ without citing any axioms.

a) Is $\sim$ **reflexive**? If so, prove it; otherwise, give a counterexample and explanation.

(b) Is $\sim$ **symmetric**? If so, prove it; otherwise, give a counterexample and explanation.

(c) Is $\sim$ **transitive**? If so, prove it; otherwise, give a counterexample and explanation.

Q8. [5 points]

(a) How many elements are there in $\mathbb{Z}_7$? List them all.

(b) Among the elements of $\mathbb{Z}_7$, find all $[x] \in \mathbb{Z}_7$ that satisfy $([x] \odot [x]) \oplus [x] = [2]$. Show all your work to justify your answer!

(c) Does $[10]$ have a multiplicative inverse in $\mathbb{Z}_7$? If so, find $[10]^{-1}$ and show that it is a multiplicative inverse. Your answer should be in terms of one of the canonical representatives.

Q9. [6 points] Consider the following subsets of real numbers:

$$A = \left\{ \frac{7n}{n+1} : n \in \mathbb{N} \right\} \qquad B = \left\{ 7 - \frac{6}{x} : x \in \mathbb{R}_{>0} \right\}$$

(a) Explain how and why the Completeness Axiom allows us to conclude that $\sup(A)$ exists.

(b) If it exists, find $\sup(B)$ and prove that it is a least upper bound for B. Otherwise, clearly justify why $\sup(B)$ does not exist.

(c) Does $\min(A)$ exist? If so, find it and justify your answer. If not, fully justify your answer.

(d) Does $\max(B)$ exist? If so, find it and justify your answer. If not, fully justify your answer.

Q10. [7 points] Consider the functions below:

$$f : (\mathbb{R}_{>0} \times \mathbb{R}_{>0}) \to \mathbb{R}_{>0} \qquad\qquad g : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \times \mathbb{R}_{>0}$$

$$f(x,y) = 5xy \qquad\qquad g(z) = \left(\tfrac{1}{5}, z\right)$$

(a) Is f injective? If so prove it; otherwise provide a concrete counterexample and briefly explain.

(b) Is f surjective? If so prove it; otherwise provide a concrete counterexample and briefly explain.

(c) Is g injective? If so prove it; otherwise provide a concrete counterexample and briefly explain.

(d) Is g surjective? If so prove it; otherwise provide a concrete counterexample and briefly explain.

(e) Is g a left inverse of f ? Justify your answer.

(f) Is g a right inverse of f ? Justify your answer.

Q11. [**4 points**] Find $\lim_{n\to\infty} \frac{7n}{n+55}$.

Rigorously prove your answer using the formal definition of a limit.
