MAT 1362 – Fall 2021

Midterm Exam

Professor: Alistair Savage

Your solutions should be submitted through Brightspace in ***.pdf, .jpg, or .png format****. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam has a possible oral component. You may be contacted after the test to arrange a Zoom meeting to explain your solutions. If you are contacted, these explanations are a part of your midterm test, and will be taken into account when determining your grade.

This exam ends at 11:15pm. You may not write anything on your pages after this time. You will then have until 11:25pm to scan and submit your solutions on Brightspace.

Question 1 **(4 pts)**. For this question, you do not need to justify your answers.

(a) State the *well-ordering principle.*
(b) State the *binomial theorem* for integers. Remember to quantify your parameters. That is, for each variable appearing in the theorem, indicate what possible values it can take.
(c) State the axiom of the *multiplicative identity* for the integers.
(d) Compute

$$\big([4598] \oplus [341]\big) \odot [7077]$$

in $\mathbb{Z}_7$. Write your answer in the standard form [0], [1], [2], [3], [4], [5], or [6].

Question 2 **(6 pts)**. Consider the following statement:

$$(\star) \qquad (\forall a, b, c \in \mathbb{N})\ a < b \implies a < bc.$$

(a) Prove the statement $(\star)$.
(b) State the converse to $(\star)$. Is the converse true? Remember to justify your answer.
(c) State the contrapositive of $(\star)$. Is the contrapositive true? Remember to justify your answer.
(d) State the negation of $(\star)$. Is the negation true? Remember to justify your answer.

Question 3 **(4 pts)**. Consider the sequence $(a_k)_{k=1}^{\infty}$ defined recursively as follows:

$$a_1 = 6, \qquad a_{k+1} = 21 + a_k - 2a_k^4 \quad \text{for } k \in \mathbb{N}.$$

Prove that a_k is divisible by 3 for all $k \in \mathbb{N}$.

Question 4 **(5 pts)**.

(a) Suppose A and B are nonempty sets. Prove that

$$\big(A \times B \subseteq B \times A\big) \iff A = B.$$

(b) Does the above double implication hold for *all* sets A or B (dropping the requirement that they be nonempty)? Remember to justify your answer.

QUESTION 5 **(5 pts)**. Consider the relation $\sim$ on $\mathbb{N}$ given by

$$a \sim b \iff a + b \text{ is divisible by } 2.$$

For each of the following, remember to justify your answers.

(a) Is $\sim$ reflexive?
(b) Is $\sim$ symmetric?
(c) Is $\sim$ transitive?
(d) Is $\sim$ an equivalence relation?