MAT 1362 – Fall 2021

Midterm Exam – Solutions

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Your solutions should be submitted through Brightspace in ***.pdf, .jpg, or .png format****. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam has a possible oral component. You may be contacted after the test to arrange a Zoom meeting to explain your solutions. If you are contacted, these explanations are a part of your midterm test, and will be taken into account when determining your grade.

This exam ends at 11:15pm. You may not write anything on your pages after this time. You will then have until 11:25pm to scan and submit your solutions on Brightspace.

QUESTION 1 **(4 pts)**. For this question, you do not need to justify your answers.

(a) State the *well-ordering principle*.

Solution: Every nonempty subset of $\mathbb{N}$ has a smallest element.

(b) State the *binomial theorem* for integers. Remember to quantify your parameters. That is, for each variable appearing in the theorem, indicate what possible values it can take.

Solution: If $a, b \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, then

$$(a+b)^k = \sum_{m=0}^{k} \binom{k}{m} a^m b^{k-m}.$$

(c) State the axiom of the *multiplicative identity* for the integers.

Solution: There exists an integer 1 such that $1 \neq 0$ and $a \cdot 1 = a$ for all $a \in \mathbb{Z}$.

(d) Compute

$$\big([4598] \oplus [341]\big) \odot [7077]$$

in $\mathbb{Z}_7$. Write your answer in the standard form [0], [1], [2], [3], [4], [5], or [6].

Solution: We have

$$\big([4598] \oplus [341]\big) \odot [7077] = \big([4598] \oplus [341]\big) \odot [0] = [0].$$

QUESTION 2 **(6 pts)**. Consider the following statement:

$$(\star) \qquad (\forall a, b, c \in \mathbb{N})\ a < b \implies a < bc.$$

(a) Prove the statement $(\star)$.

Solution: Suppose that $a, b, c \in \mathbb{N}$ and $a < b$. Since $c \in \mathbb{N}$, we have $c \geq 1$. Thus

$$b = b \cdot 1 \leq b \cdot c.$$

Therefore, $a < b \leq bc$, which implies that $a < bc$.

(b) State the converse to $(\star)$. Is the converse true? Remember to justify your answer.

Solution: The converse is

$$(\forall a, b, c \in \mathbb{N})\ a < bc \implies a < b.$$

This is false. For example, for $a = b = 1$ and $c = 2$, we have $a = 1 < 2 = bc$, but $a = 1 \not< 1 = b$.

(c) State the contrapositive of $(\star)$. Is the contrapositive true? Remember to justify your answer.

Solution: The contrapositive is

$$(\forall a, b, c \in \mathbb{N})\ a \geq bc \implies a \geq b.$$

This is true since $(\star)$ is true, and a statement is equivalent to its contrapositive.

(d) State the negation of $(\star)$. Is the negation true? Remember to justify your answer.

Solution: Since the negation of $A \implies B$ is (A and $\neg B$), the negation of $(\star)$ is

$$(\exists a, b, c \in \mathbb{N} \text{ such that})\ a < b \text{ and } a \geq bc.$$

This is false, since $(\star)$ is true.

QUESTION 3 **(4 pts)**. Consider the sequence $(a_k)_{k=1}^{\infty}$ defined recursively as follows:

$$a_1 = 6, \qquad a_{k+1} = 21 + a_k - 2a_k^4 \quad \text{for } k \in \mathbb{N}.$$

Prove that a_k is divisible by 3 for all $k \in \mathbb{N}$.

Solution: We prove the result by induction on k.

Base case: Since $a_1 = 6 = 2 \cdot 3$, the assertion is true for $k = 1$.

Induction step: Suppose a_k is divisible by 3 for some $k \in \mathbb{N}$. Thus, there exists $n \in \mathbb{Z}$ such that $a_k = 3n$. Then

$$a_{k+1} = 21 + a_k - 2a_k^4 = 21 + 3n - 2 \cdot (3n)^4 = 21 + 3n - 162n^4 = 3(7 + n - 54n^4).$$

Hence a_{k+1} is divisible by 3. This completes the proof of the induction step.

QUESTION 4 **(5 pts)**.

(a) Suppose A and B are nonempty sets. Prove that

$$\big(A \times B \subseteq B \times A\big) \iff A = B.$$

Solution: If $A = B$, then $A \times B = B \times A$. So the reverse implication holds. Now suppose that $A \times B \subseteq B \times A$. Let $a \in A$. Since B is nonempty, we can choose $b \in B$. Then

$$(a, b) \in A \times B \subseteq B \times A \implies (a, b) \in B \times A \implies a \in B.$$

Hence $A \subseteq B$. On the other hand, let $b \in B$. Since A is nonempty, we can choose $a \in A$. Then

$$(a, b) \in A \times B \subseteq B \times A \implies b \in A.$$

Hence $B \subseteq A$. Therefore $A = B$.

(b) Does the above double implication hold for *all* sets A or B (dropping the requirement that they be nonempty)? Remember to justify your answer.

Solution: No, it does not hold for all sets. For example,

$$\varnothing \times \mathbb{Z} = \varnothing = \mathbb{Z} \times \varnothing,$$

but $\varnothing \neq \mathbb{Z}$.

QUESTION 5 **(5 pts)**. Consider the relation $\sim$ on $\mathbb{N}$ given by

$$a \sim b \iff a + b \text{ is divisible by } 2.$$

For each of the following, remember to justify your answers.

(a) Is $\sim$ reflexive?

Solution: Yes, $\sim$ is reflexive. For all $a \in \mathbb{N}$, we have that $a + a = 2a$ is divisible by 2, and so $a \sim a$.

(b) Is $\sim$ symmetric?

Solution: Yes, $\sim$ is symmetric. Suppose $a, b \in \mathbb{N}$ and $a \sim b$. Then there exists $n \in \mathbb{Z}$ such that $a + b = 2n$. Hence $b + a = 2n$ is also divisible by 2. So $b \sim a$.

(c) Is $\sim$ transitive?

Solution: Yes, $\sim$ is transitive. Suppose $a, b, c \in \mathbb{N}$, $a \sim b$, and $b \sim c$. Then there exist $m, n \in \mathbb{N}$ such that $a + b = 2m$ and $b + c = 2n$. Then

$$a + c = (a + b) + (b + c) - 2b = 2m + 2n - 2b = 2(m + n - b).$$

Thus $a + c$ is divisible by 2, and so $a \sim c$.

(d) Is $\sim$ an equivalence relation?

Solution: Yes, $\sim$ is an equivalence relation since it is reflexive, symmetric, and transitive.