MAT 1362 – Fall 2021

Final Exam

Professor: Alistair Savage

*Your solutions should be submitted through Brightspace in **.pdf, .jpg, or .png format**. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam has a possible oral component. You may be contacted after the test to arrange a Zoom meeting to explain your solutions. If you are contacted, these explanations are a part of your final exam, and will be taken into account when determining your grade.

This exam ends at 12:30pm. You may not write anything on your pages after this time. You will then have until 12:40pm to scan and submit your solutions on Brightspace. You must remain on camera until your work is submitted.

If you wish to leave the exam early, you must request permission by sending a private message to the instructor in the Zoom chat. Once permission is granted, you may not write anything further on your pages, and you have 10 minutes to scan and submit your solutions.

Question 1 **(4 pts)**. For this question, you do not need to justify your answers.

(a) Give the definition of the ordering on the integers. In other words, complete the following definition: "For $a, b \in \mathbb{Z}$, we write $a < b$ (and say a is *less than* b) if and only if…".
(b) State the *cancellation* axiom for the integers.
(c) Let $r \in \mathbb{R}_{>0}$. Give the precise definition of the *square root* of r (the definition that relies on the completeness axiom for the real numbers).
(d) Define the *symmetric difference* $A \Delta B$ for two sets A and B.

Question 2 **(5 pts)**.

(a) Consider the following statement:

$$(x < 2 \text{ or } x > 2) \implies x \neq 2. \tag{$\star$}$$

For each of the following, simplify your answer as much as possible. In particular, your answer should not include the word "not" or the symbol $\neg$.
  (i) State the converse of ($\star$).
  (ii) State the contrapositive of ($\star$).
  (iii) State the negation of ($\star$).

(b) Consider the following statement:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } x \cdot y = 1. \tag{$\dagger$}$$

  (i) Is the statement ($\dagger$) true or false? Justify your answer.
  (ii) State the negation of the statement ($\dagger$)

Question 3 **(4 pts)**. Using induction, prove that

$$\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \frac{n(n+1)}{2} \quad \text{for all } n \in \mathbb{N}.$$

Remember to clearly state your induction hypothesis.

Question 4 **(4 pts)**. Prove that, for all sets A, B, C,

$$A - (B \cap C) = (A - B) \cup (A - C).$$

QUESTION 5 **(4 pts)**. Consider the sets $A = \{10n + 8 : n \in \mathbb{Z}\}$ and $B = \{5m + 3 : m \in \mathbb{Z}\}$.

(a) Prove that $A \subseteq B$.
(b) Is it true that $A = B$? Remember to fully justify your answer.

QUESTION 6 **(5+1 pts)**. Let

$$A = \{(x, y) : x, y \in \mathbb{R},\ (x, y) \neq (0, 0)\} \subseteq \mathbb{R} \times \mathbb{R}$$

and let $\sim$ be the equivalence relation on A defined by

$$(x, y) \sim (w, z) \iff \big(\exists c \in \mathbb{R} \text{ such that } (x, y) = (cw, cz)\big).$$

(a) **(5 pts)** Show that $\sim$ is an equivalence relation.
(b) **(Bonus: 1 pt)** Let X be the set of equivalence classes of A. Is the function

$$f\colon X \to \mathbb{R}, \quad f([(x, y)]) = x + y,$$

well defined? *Note*: In order to get this bonus point, you must justify your answer. A guess of 'yes' or 'no' without justification will not receive any points.

QUESTION 7 **(4 pts)**.

(a) State Fermat's Little Theorem.
(b) Is $x^7 + 14x^5 + 7x^2 + 6x - 21$ divisible by 7 for all $x \in \mathbb{Z}$? Justify your answer using modular arithmetic.

QUESTION 8 **(7 pts)**. Consider the set

$$B = \left\{3 - \frac{4}{x} : x \in \mathbb{R},\ x \geq 2\right\}.$$

For this question, you may use the fact that the set $\{x \in \mathbb{R} : x \geq 2\}$ is not bounded above.

(a) Find the smallest element (i.e. minimum) of B or justify that it does not exist.
(b) Find the infimum of B or justify that it does not exist.
(c) Find the supremum of B or justify that it does not exist.
(d) Find the largest element (i.e. maximum) of B or justify that it does not exist.

QUESTION 9 **(7 pts)**. Consider the function

$$f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad f(x) = |x - 1|.$$

(a) Is f injective? Justify your answer.
(b) Is f surjective? Justify your answer.
(c) Does f have a left inverse? Justify your answer.
(d) Does f have a right inverse? Justify your answer.
(e) Does f has a two-sided inverse? Justify your answer.

QUESTION 10 **(4 pts)**.

(a) State the definition of a limit of a sequence of real numbers. In other words, if $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}$, state precisely what it means for the sequence to converge to $L \in \mathbb{R}$.
(b) Find the limit

$$\lim_{n \to \infty} \left(3 - \frac{3}{7n^2 + 2}\right).$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences, or the arithmetic of limits.