MAT 1362 – Fall 2021

Final Exam – Solutions

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*Your solutions should be submitted through Brightspace in **.pdf, .jpg, or .png format**. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam has a possible oral component. You may be contacted after the test to arrange a Zoom meeting to explain your solutions. If you are contacted, these explanations are a part of your final exam, and will be taken into account when determining your grade.

This exam ends at 12:30pm. You may not write anything on your pages after this time. You will then have until 12:40pm to scan and submit your solutions on Brightspace. You must remain on camera until your work is submitted.

If you wish to leave the exam early, you must request permission by sending a private message to the instructor in the Zoom chat. Once permission is granted, you may not write anything further on your pages, and you have 10 minutes to scan and submit your solutions.

QUESTION 1 **(4 pts)**. For this question, you do not need to justify your answers.

(a) Give the definition of the ordering on the integers. In other words, complete the following definition: "For $a, b \in \mathbb{Z}$, we write $a < b$ (and say a is *less than* b) if and only if. . . ".

Solution: For $a, b \in \mathbb{Z}$, we write $a < b$ (and say a is *less than* b) if and only if $b - a \in \mathbb{N}$.

(b) State the *cancellation* axiom for the integers.

Solution: If $a, b, c \in \mathbb{Z}$, $a \cdot b = a \cdot c$, and $a \neq 0$, then $b = c$.

(c) Let $r \in \mathbb{R}_{>0}$. Give the precise definition of the *square root* of r (the definition that relies on the completeness axiom for the real numbers).

Solution: $\sqrt{r} := \sup\{x \in \mathbb{R} : x^2 < r\}$.

(d) Define the *symmetric difference* $A \Delta B$ for two sets A and B.

Solution: $A \Delta B := (A - B) \cup (B - A)$.

QUESTION 2 **(5 pts)**.

(a) Consider the following statement:

$$(x < 2 \text{ or } x > 2) \implies x \neq 2. \tag{$\star$}$$

For each of the following, simplify your answer as much as possible. In particular, your answer should not include the word "not" or the symbol $\neg$.

(i) State the converse of ($\star$).
(ii) State the contrapositive of ($\star$).
(iii) State the negation of ($\star$).

Solution:
(i) $x \neq 2 \implies (x < 2 \text{ or } x > 2)$
(ii) $x = 2 \implies (x \geq 2 \text{ and } x \leq 2)$
(iii) $(x < 2 \text{ or } x > 2)$ and $x = 2$

(b) Consider the following statement:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } x \cdot y = 1. \tag{$\dagger$}$$

(i) Is the statement (†) true or false? Justify your answer.
(ii) State the negation of the statement (†)

Solution:
(i) The statement is false since, for $x = 0$, we have $x \cdot y = 0 \cdot y = 0 \neq 1$ for all $y \in \mathbb{R}$.
(ii) $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R},\ x \cdot y \neq 1$

Question 3 **(4 pts)**. Using induction, prove that

$$\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \frac{n(n+1)}{2} \quad \text{for all } n \in \mathbb{N}.$$

Remember to clearly state your induction hypothesis.

Solution: *Base case*: When $n = 1$, we have

$$\sum_{k=1}^{1} (-1)^k k^2 = (-1)^1 \cdot 1^2 = -1 = (-1)^1 \frac{1(1+1)}{2}.$$

Thus the result holds for $n = 1$.

Induction step: Suppose that, for some $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}.$$

Then

$$\begin{aligned}
\sum_{k=1}^{n+1} (-1)^k k^2 &= \sum_{k=1}^{n} (-1)^k k^2 + (-1)^{n+1}(n+1)^2 \\
&= (-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n+1)^2 \\
&= (-1)^n (n+1) \left(\frac{n}{2} - (n+1) \right) \\
&= (-1)^n (n+1) \frac{-n-2}{2} \\
&= (-1)^{n+1} \frac{(n+1)(n+2)}{2}.
\end{aligned}$$

This completes the proof of the induction step.

Question 4 **(4 pts)**. Prove that, for all sets A, B, C,

$$A - (B \cap C) = (A - B) \cup (A - C).$$

Solution: We have

$$\begin{aligned}
x \in A - (B \cap C) &\iff x \in A \text{ and } \neg(x \in B \cap C) \\
&\iff x \in A \text{ and } \neg(x \in B \text{ and } x \in C) \\
&\iff x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\
&\iff (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\
&\iff x \in A - B \text{ or } x \in A - C \\
&\iff x \in (A - B) \cup (A - C).
\end{aligned}$$

QUESTION 5 **(4 pts)**. Consider the sets $A = \{10n + 8 : n \in \mathbb{Z}\}$ and $B = \{5m + 3 : m \in \mathbb{Z}\}$.

(a) Prove that $A \subseteq B$.

Solution: Let $x \in A$. By the definition of A, there exists $n \in Z$ such that $x = 10n + 8$. Then we have

$$x = 10n + 8 = 5 \cdot (2n) + 5 + 3 = 5 \cdot (2n + 1) + 3 = 5m + 3 \text{ for } m = 2n + 1.$$

We have $m = 2n + 1 \in \mathbb{Z}$ since $1, 2, n \in \mathbb{Z}$. Hence $x = 5m + 3 \in B$, and so $A \subseteq B$.

(b) Is it true that $A = B$? Remember to fully justify your answer.

Solution: No, $A \neq B$. For example, $13 = 5 \cdot 2 + 3 \in B$, but $13 \notin A$. We prove that $13 \notin A$ by contradiction. If $13 \in A$, then there exists $n \in \mathbb{Z}$ such that

$$13 = 10n + 8 \implies 10n = 5 \implies 2n = 1 \implies n = \frac{1}{2}.$$

Since $\frac{1}{2} \notin \mathbb{Z}$, this is a contradiction.

QUESTION 6 **(5+1 pts)**. Let

$$A = \{(x, y) : x, y \in \mathbb{R},\ (x, y) \neq (0, 0)\} \subseteq \mathbb{R} \times \mathbb{R}$$

and let $\sim$ be the equivalence relation on A defined by

$$(x, y) \sim (w, z) \iff \big(\exists c \in \mathbb{R} \text{ such that } (x, y) = (cw, cz)\big).$$

(a) **(5 pts)** Show that $\sim$ is an equivalence relation.

Solution: *Reflexivity*: Suppose $(x, y) \in A$. Then $(x, y) = (1 \cdot x, 1 \cdot y)$, and so $(x, y) \sim (x, y)$.

Symmetry: Suppose $(x, y), (w, z) \in A$ satisfy $(x, y) \sim (w, z)$. Then there exists $c \in \mathbb{R}$ such that $(x, y) = (cw, cz)$. Since $cw = x \neq 0$, we have $c \neq 0$. Then $(w, z) = \big(\frac{1}{c} \cdot x, \frac{1}{c} \cdot y\big)$, and so $(w, z) \sim (x, y)$.

Transitivity: Suppose $(x, y), (w, z), (a, b) \in A$ satisfy $(x, y) \sim (w, z)$ and $(w, z) \sim (a, b)$. Then there exist $c, d \in \mathbb{R}$ such that

$$(x, y) = (cw, cz) \text{ and } (w, z) = (da, db).$$

Therefore

$$(x, y) = \big((cd)a, (cd)b\big),$$

and so $(x, y) \sim (a, b)$.

(b) **(Bonus: 1 pt)** Let X be the set of equivalence classes of A. Is the function

$$f \colon X \to \mathbb{R}, \quad f([(x, y)]) = x + y,$$

well defined? *Note*: In order to get this bonus point, you must justify your answer. A guess of 'yes' or 'no' without justification will not receive any points.

Solution: No, it is not well defined. For example $[(1, 1)] = [(2, 2)]$, but

$$f([(1, 1)]) = 1 + 1 = 2 \neq 4 = 2 + 2 = f([(2, 2)]).$$

QUESTION 7 **(4 pts)**.

(a) State Fermat's Little Theorem.

Solution: If $m \in \mathbb{Z}$ and p is prime, then

$$m^p \equiv m \mod p.$$

(b) Is $x^7 + 14x^5 + 7x^2 + 6x - 21$ divisible by 7 for all $x \in \mathbb{Z}$? Justify your answer using modular arithmetic.

Solution: Fix the modulus 7. By Fermat's Little Theorem, we have $x^7 \equiv x$. Thus

$$x^7 + 14x^5 + 7x^2 + 6x - 21 \equiv 14x^5 + 7x^2 + 7x - 21 \equiv 0x^5 + 0x^2 + 0x - 0 \equiv 0.$$

Thus $x^7 + 14x^5 + 7x^2 + 6x - 21$ is divisible by 7.

QUESTION 8 **(7 pts)**. Consider the set

$$B = \left\{ 3 - \frac{4}{x} : x \in \mathbb{R},\ x \geq 2 \right\}.$$

For this question, you may use the fact that the set $\{x \in \mathbb{R} : x \geq 2\}$ is not bounded above.

(a) Find the smallest element (i.e. minimum) of B or justify that it does not exist.

Solution: We will show that $\min(B) = 1$. First note that $1 = 3 - \frac{4}{2} \in B$. Furthermore, for $x \in \mathbb{R}$, we have

$$x \geq 2 \implies \frac{1}{x} \leq \frac{1}{2} \implies -\frac{4}{x} \geq -\frac{4}{2} = -2 \implies 3 - \frac{4}{x} \geq 1.$$

Thus, all elements of B are greater than or equal to 1.

(b) Find the infimum of B or justify that it does not exist.

Solution: Since B has a smallest element, we have $\inf(B) = \min(B) = 1$.

(c) Find the supremum of B or justify that it does not exist.

Solution: We will show that $\sup(B) = 3$. First we show that 3 is an upper bound for B. Indeed, for $x \in \mathbb{R}$,

$$x \geq 2 \implies x > 0 \implies \frac{1}{x} > 0 \implies -\frac{4}{x} < 0 \implies 3 - \frac{4}{x} < 3.$$

It remains to show that 3 is the *least* upper bound for B. Towards a contradiction, suppose that $b < 3$ is an upper bound for B. Then, for all $x \in \mathbb{R}$, $x \geq 2$, we have

$$\begin{aligned} & 3 - \frac{4}{x} \leq b \\ \implies & \frac{4}{x} \geq 3 - b \\ \implies & \frac{x}{4} \leq \frac{1}{3-b} && \text{(since } 3 - b > 0\text{)} \\ \implies & x \leq \frac{4}{3-b}. \end{aligned}$$

But this means that $\frac{4}{3-b}$ is an upper bound for $\{x \in \mathbb{R} : x \geq 2\}$, contradicting the fact that this set is not bounded above.

(d) Find the largest element (i.e. maximum) of B or justify that it does not exist.

Solution: If B had a largest element, it would have to be $\sup(B) = 3$. However, as we saw above, all elements of B are strictly less than 3. So $3 \notin B$. Hence B has no largest element.

QUESTION 9 **(7 pts)**. Consider the function

$$f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad f(x) = |x-1|.$$

(a) Is f injective? Justify your answer.

Solution: No, f is not injective. For example

$$f(0) = |0-1| = 1 = |2-1| = f(2).$$

(b) Is f surjective? Justify your answer.

Solution: Yes, f is surjective. For any $y \in \mathbb{R}_{\geq 0}$, we have $y+1 \in \mathbb{R}_{\geq 0}$ and

$$f(y+1) = |(y+1)-1| = |y| = y \qquad (\text{since } y \geq 0).$$

(c) Does f have a left inverse? Justify your answer.

Solution: No. A function has a left inverse if and only if it is injective. Since f is not injective, it does not have a left inverse.

(d) Does f have a right inverse? Justify your answer.

Solution: Yes. A function has a right inverse if and only if it is surjective. Since f is surjective, it has a right inverse.

(e) Does f has a two-sided inverse? Justify your answer.

Solution: No. Since f does not have a left inverse, it does not have a two-sided inverse.

QUESTION 10 **(4 pts)**.

(a) State the definition of a limit of a sequence of real numbers. In other words, if $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}$, state precisely what it means for the sequence to converge to $L \in \mathbb{R}$.

Solution: The sequence $(x_k)_{k=1}^{\infty}$ converges to L if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |x_n - L| < \varepsilon.$$

(b) Find the limit

$$\lim_{n\to\infty}\left(3-\frac{3}{7n^2+2}\right).$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences, or the arithmetic of limits.

Solution: We will prove that the limit is 3. Note that, for $n \in \mathbb{N}$, we have

$$\left|-\frac{3}{7n^2+2}\right| = \frac{3}{7n^2+2} \leq \frac{3}{7n^2} \leq \frac{3}{7n}.$$

Let $\varepsilon > 0$. Since $\mathbb{N}$ has no upper bound, we can choose $N \in \mathbb{N}$ such that

$$N > \frac{3}{7\varepsilon}.$$

Then, for $n \geq N$, we have

$$\left|\left(3-\frac{3}{7n^2+2}\right)-3\right| = \left|-\frac{3}{7n^2+2}\right| \leq \frac{3}{7n} \leq \frac{3}{7N} < \varepsilon.$$