MAT 1362 – Fall 2020

Midterm Exam – Solutions

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Your solutions should be submitted through Brightspace as a ***single pdf file****. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam ends at 11:15am. You may not write anything on your pages after this time. You will then have until 11:25am to scan and submit your solutions on Brightspace.

QUESTION 1 **(4 pts)**. Using induction, prove that

$$\sum_{n=5}^{k} \frac{1}{n^2-5n+6} = \frac{k-4}{2k-4} \quad \text{for all } k \in \mathbb{Z},\ k \geq 5.$$

Solution: *Base case*: When $k=5$, we have

$$\sum_{n=5}^{5} \frac{1}{n^2-5n+6} = \frac{1}{5^2-5\cdot 5+6} = \frac{1}{6} = \frac{5-6}{2\cdot 5-4}.$$

Thus, the given equality holds for $k=5$.

Induction step: Suppose the given equality holds for some $k \geq 5$. Then we have

$$\begin{aligned}
\sum_{n=5}^{k+1} \frac{1}{n^2-5n+6} &= \sum_{n=5}^{k} \frac{1}{n^2-5n+6} + \frac{1}{(k+1)^2-5(k+1)+6} \\
&= \frac{k-4}{2k-4} + \frac{1}{k^2-3k+2} \\
&= \frac{k-4}{2(k-2)} + \frac{1}{(k-1)(k-2)} \\
&= \frac{(k-4)(k-1)+2}{2(k-1)(k-2)} \\
&= \frac{(k-3)(k-2)}{2(k-1)(k-2)} \\
&= \frac{k-3}{2k-2} \\
&= \frac{(k+1)-4}{2(k+1)-4}.
\end{aligned}$$

This completes the proof of the induction step.

QUESTION 2 **(4 pts)**.

(a) Let P be the statement

$$(\exists n \in \mathbb{Z} \text{ such that})(\forall m \in \mathbb{N})\ n+3m=0$$

(i) Is the statement P true or false? Remember to justify your answer.

Solution: The statement is false. For example, if $n=0$, then taking $m=1$ gives $n+3m=3\neq 0$. On the other hand, if $n\neq 0$, then taking $m=0$ gives $n+3m=n\neq 0$.

(ii) Write the negation of the statement P. Simplify your answer as much as possible; the symbol $\neg$ should not appear in your answer.

Solution: The negation of P is

$$(\forall n \in \mathbb{Z})(\exists m \in \mathbb{N} \text{ such that) } n + 3m \neq 0.$$

(b) Let Q be the statement

$$(\forall n, m \in \mathbb{Z})\ (n^2 < 0 \implies m \text{ is even}).$$

(i) Is the statement Q true or false? Remember to justify your answer.

Solution: The statement Q is true. For all $n \in \mathbb{Z}$, the statement $n^2 < 0$ is false, and hence the implication is true.

(ii) Write the negation of the statement Q. Simplify your answer as much as possible; the symbol $\neg$ should not appear in your answer.

Solution: The negation of Q is

$$(\exists n, m \in \mathbb{Z} \text{ such that) } (n^2 < 0 \text{ and } m \text{ is odd}).$$

Question 3 **(3 pts)**. Prove that

$$(A \cap B) \times C = (A \times C) \cap (B \times C)$$

for all sets A, B, and C.

Solution: We have

$$\begin{aligned}
(x, y) \in (A \cap B) \times C &\iff x \in A \cap B \text{ and } y \in C \\
&\iff x \in A \text{ and } x \in B \text{ and } y \in C \\
&\iff (x \in A \text{ and } y \in C) \text{ and } (x \in B \text{ and } y \in C) \\
&\iff (x, y) \in A \times C \text{ and } (x, y) \in B \times C \\
&\iff (x, y) \in (A \times C) \cap (B \times C).
\end{aligned}$$

Thus $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

Question 4 **(5 pts)**. Consider the relation on $\mathbb{N} \times \mathbb{N}$ given by

$$(a, b) \sim (c, d) \iff b - d = 4(a - c).$$

Prove that $\sim$ is an equivalence relation.

Solution: *Reflexivity*: Suppose $(a, b) \in \mathbb{N} \times \mathbb{N}$. Since $b - b = 0 = 4(a - a)$, we see that $(a, b) \sim (a, b)$ Hence $(a, b) \sim (a, b)$.

Symmetry: Suppose $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$ and $(a, b) \sim (c, d)$. Then $b - d = 4(a - c)$. Multiplying both sides by -1, we see that $d - b = 4(c - a)$. Therefore $(c, d) \sim (a, b)$.

Transitivity: Suppose $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$, $(a, b) \sim (c, d)$, and $(c, d) \sim (e, f)$. Thus

$$b - d = 4(a - c) \quad \text{and} \quad d - f = 4(c - e).$$

Hence

$$b - f = (b - d) + (d - f) = 4(a - c) + 4(c - e) = 4(a - e),$$

and so $(a, b) \sim (e, f)$.

QUESTION 5 **(2 pts)**.

(a) Compute

$$[81] \odot ([72] \oplus [575])$$

in $\mathbb{Z}_5$. Write your answer in the standard form: [0], [1], [2], [3], or [4].

Solution: We have

$$[81] \odot ([72] \oplus [575]) = [1] \odot ([2] \oplus [0]) = [1] \odot [2] = [2].$$

(b) What is the additive inverse of [4] in $\mathbb{Z}_{11}$? Write your answer in the standard form: [0], [1],...,[9], or [10]. Remember to justify your answer.

Solution: Since $[4] \oplus [7] = [11] = [0]$, we see that the additive inverse of [4] is [7].