MAT 1362 – Fall 2020

Final Exam

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Your solutions should be submitted through Brightspace as a ***single pdf file****. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam ends at 5:00pm. You may not write anything on your pages after this time. You will then have until 5:10pm to scan and submit your solutions on Brightspace.

If you wish to leave the exam early, you must request permission in the Zoom chat. Once permission is granted, you may not write anything further on your pages, and you have 10 minutes to scan and submit your solutions.

Question 1 **(4 pts)**. For this question, you do not need to justify your answers.

(a) Suppose $f: A \to B$ is a function. Give the definition of a *left inverse* of f.
(b) Suppose $\sim$ is an equivalence relation on a nonempty set A. Give the definition of the *equivalence class* of an element $a \in A$.
(c) State *Euclid's lemma*.
(d) For which values of $a \in \mathbb{N}$ is $\sqrt{a}$ irrational?

Question 2 **(4 pts)**. Consider the sequence $(x_k)_{k=1}^{\infty}$ defined recursively as follows:

$$x_1 = -12, \qquad x_{k+1} = 3x_k^2 + 16 \text{ for } k \in \mathbb{N}.$$

Prove that x_k is divisible by 4 for all $k \in \mathbb{N}$.

Question 3 **(5 pts)**.

(a) For each of the following statements, indicate whether the statement is true or false. You do not need to justify your answers. Your grade on this part will be based on the number of correct answers minus the number of incorrect answers. (You may also leave an answer blank, in which case you receive zero points for that answer.) You cannot receive a negative grade.
  (i) $(\forall n \in \mathbb{Z})\ (\exists x \in \mathbb{R} \text{ such that})\ x < n$.
  (ii) $(\exists x \in \mathbb{R} \text{ such that})\ (\forall n \in \mathbb{Z})\ x < n$.
  (iii) (3 is an even number) $\implies$ (4 is an odd number).
  (iv) $(\forall A \subseteq \mathbb{N})\ (\exists!\, a \in A \text{ such that})\ (\forall b \in A)\ a \le b$.
(b) Suppose $f: \mathbb{R} \to \mathbb{R}$ is a function. Write the negation of the following implication:

$$(\forall x \in \mathbb{R}_{>0})\ (\exists y, z \in \mathbb{Z} \text{ such that})\ f(x)f(y) > 0 \implies f(z) \neq f(x)$$

Simplify your answer as much as possible. In particular, the symbol $\neg$ should not appear in your final answer.
(c) Suppose $A, B, C \subseteq \mathbb{R}$, and let P be the statement

$$\text{If } x \in A \text{ and } y \in B, \text{ then } xy \in C.$$

  (i) Write the converse of P. Simplify your answer as much as possible. In particular, the symbol $\neg$ should not appear in your final answer.
  (ii) Write the contrapositive of P. Simplify your answer as much as possible. In particular, the symbol $\neg$ should not appear in your final answer.

QUESTION 4 **(4 pts)**. Using induction, prove that

$$\sum_{k=2}^{n} \frac{1}{k^2+7k+12} = \frac{n-1}{5n+20} \quad \text{for all } n \in \mathbb{Z},\ n \geq 2.$$

QUESTION 5 **(3 pts)**. Suppose A and B are subsets of a set X. Show that

$$(A \cap B)^{\mathsf{C}} = A^{\mathsf{C}} \cup B^{\mathsf{C}}.$$

QUESTION 6 **(4 pts)**. Consider the following subsets of $\mathbb{R} \times \mathbb{R}$:

$$A = \{(x, x+2) : x \in \mathbb{R}\}, \qquad B = \{(2y-1, 1+2y) : y \in \mathbb{R}\}.$$

Prove that $A = B$.

QUESTION 7 **(5 pts)**. Let

$$X = \{x \in \mathbb{R} : x \neq 0\}$$

be the set of nonzero real numbers. Let $\sim$ be the relation on X defined by

$$x \sim y \iff xy^{-1} \in \mathbb{Z}.$$

(a) Is $\sim$ reflexive?
(b) Is $\sim$ symmetric?
(c) Is $\sim$ transitive?
(d) Is $\sim$ an equivalence relation?

Remember to justify your answers.

QUESTION 8 **(4 pts)**.

(a) Fix $n \in \mathbb{N}$. State the definition of *congruence modulo* n. More precisely, complete the following sentence: "For $x, y \in \mathbb{Z}$, we say that x and y are congruent modulo n if and only if..."
(b) Suppose that $m, n \in \mathbb{N}$ and that m divides n. For $x, y \in \mathbb{Z}$, show that

$$(x \equiv y \mod n) \implies (x \equiv y \mod m).$$

QUESTION 9 **(7 pts)**. Consider the set

$$A = \left\{5 + \frac{2}{n} : n \in \mathbb{N}\right\}.$$

(a) Find the largest element (i.e. maximum) of A or show that it does not exist. In either case, remember to justify your answer.
(b) Find the supremum of A or show that it does not exist. In either case, remember to justify your answer.
(c) Find the infimum of A or show that it does not exist. In either case, remember to justify your answer.
(d) Find the smallest element (i.e. minimum) of A or show that it does not exist. In either case, remember to justify your answer.

QUESTION 10 **(4 pts)**. Suppose A and B are nonempty sets, and consider the function

$$f\colon A \times B \to A, \quad f(a,b) = a \quad \text{for } (a,b) \in A \times B.$$

(a) Is f surjective? Remember to justify your answer.
(b) When is f injective? More precisely, give conditions on the sets A and/or B that are satisfied if and only if f is injective.

QUESTION 11 **(4 pts)**.

(a) State the definition of a limit of a sequence of real numbers. In other words, if $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}$, state precisely what it means for the sequence to converge to $L \in \mathbb{R}$.

(b) What is

$$\lim_{n\to\infty} \left(4 + \frac{2n-1}{3n^2+5}\right) ?$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences, or the arithmetic of limits.