MAT 1362 – Fall 2020

Final Exam – Solutions

Professor: Alistair Savage

Your solutions should be submitted through Brightspace as a ***single pdf file****. It is your responsibility to make sure that your handwriting is legible and that your scan is of high enough quality that it can be easily read. You should always justify your answer, unless otherwise specified.*

This exam ends at 5:00pm. You may not write anything on your pages after this time. You will then have until 5:10pm to scan and submit your solutions on Brightspace.

If you wish to leave the exam early, you must request permission in the Zoom chat. Once permission is granted, you may not write anything further on your pages, and you have 10 minutes to scan and submit your solutions.

QUESTION 1 **(4 pts)**. For this question, you do not need to justify your answers.

(a) Suppose $f\colon A \to B$ is a function. Give the definition of a *left inverse* of f.

Solution: A left inverse of f is a function $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$.

(b) Suppose $\sim$ is an equivalence relation on a nonempty set A. Give the definition of the *equivalence class* of an element $a \in A$.

Solution: The equivalence class of $a \in A$ is

$$[a] = \{b \in A : b \sim a\}.$$

(c) State *Euclid's lemma*.

Solution: If p is a prime number, $m, n \in \mathbb{N}$, and $p|mn$, then $p|m$ or $p|n$.

(d) For which values of $a \in \mathbb{N}$ is $\sqrt{a}$ irrational?

Solution: The real number $\sqrt{a}$ is irrational if and only if a is not a perfect square.

QUESTION 2 **(4 pts)**. Consider the sequence $(x_k)_{k=1}^{\infty}$ defined recursively as follows:

$$x_1 = -12, \qquad x_{k+1} = 3x_k^2 + 16 \text{ for } k \in \mathbb{N}.$$

Prove that x_k is divisible by 4 for all $k \in \mathbb{N}$.

Solution: We prove the result by induction on k.

Base case: Since $x_1 = -12 = (-3)4$, the assertion is true for $k = 1$.

Induction step: Suppose x_k is divisible by 4 for some $k \in \mathbb{N}$. Thus there exists $n \in \mathbb{Z}$ such that $x_k = 4n$. Then

$$x_{k+1} = 3x_k^2 + 16 = 3(4n)^2 + 16 = 48n^2 + 16 = 4(12n^2 + 4).$$

Hence x_{k+1} is divisible by 4. This completes the proof of the induction step.

QUESTION 3 **(5 pts)**.

(a) For each of the following statements, indicate whether the statement is true or false. You do not need to justify your answers. Your grade on this part will be based on the number of correct answers minus the number of incorrect answers. (You may also leave an answer blank, in which case you receive zero points for that answer.) You cannot receive a negative grade.

(i) $(\forall n \in \mathbb{Z})\ (\exists x \in \mathbb{R} \text{ such that})\ x < n$.
(ii) $(\exists x \in \mathbb{R} \text{ such that})\ (\forall n \in \mathbb{Z})\ x < n$.
(iii) (3 is an even number) $\implies$ (4 is an odd number).
(iv) $(\forall A \subseteq \mathbb{N})\ (\exists!\, a \in A \text{ such that})\ (\forall b \in A)\ a \leq b$.

Solution:
(i) True.
(ii) False.
(iii) True.
(iv) True. This is the well-ordering principle.

(b) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function. Write the negation of the following implication:

$$(\forall x \in \mathbb{R}_{>0})\ (\exists y, z \in \mathbb{Z} \text{ such that})\ f(x)f(y) > 0 \implies f(z) \neq f(x)$$

Simplify your answer as much as possible. In particular, the symbol $\neg$ should not appear in your final answer.

Solution: The negation is

$$(\exists x \in \mathbb{R}_{>0} \text{ such that})\ (\forall y, z \in \mathbb{Z})\ f(x)f(y) > 0 \text{ and } f(z) = f(x).$$

(c) Suppose $A, B, C \subseteq \mathbb{R}$, and let P be the statement

If $x \in A$ and $y \in B$, then $xy \in C$.

(i) Write the converse of P. Simplify your answer as much as possible. In particular, the symbol $\neg$ should not appear in your final answer.

Solution: If $xy \in C$, then $x \in A$ and $y \in B$.

(ii) Write the contrapositive of P. Simplify your answer as much as possible. In particular, the symbol $\neg$ should not appear in your final answer.

Solution: If $xy \notin C$, then $x \notin A$ or $y \notin B$.

QUESTION 4 **(4 pts)**. Using induction, prove that

$$\sum_{k=2}^{n} \frac{1}{k^2 + 7k + 12} = \frac{n-1}{5n+20} \quad \text{for all } n \in \mathbb{Z},\ n \geq 2.$$

Solution: *Base case*: When $n = 2$, we have

$$\sum_{k=2}^{2} \frac{1}{k^2 + 7k + 12} = \frac{1}{2^2 + 7 \cdot 2 + 12} = \frac{1}{30} = \frac{2-1}{5 \cdot 2 + 20}.$$

Thus, the given equality holds for $n = 2$.

Induction step: Suppose the given equality holds for some $n \geq 2$. Then we have

$$\sum_{k=2}^{n+1} \frac{1}{k^2 + 7k + 12} = \sum_{k=2}^{n} \frac{1}{k^2 + 7k + 12} + \frac{1}{(n+1)^2 + 7(n+1) + 12}$$

$$
\begin{aligned}
&= \frac{n-1}{5n+20} + \frac{1}{n^2+9n+20} \\
&= \frac{n-1}{5(n+4)} + \frac{1}{(n+4)(n+5)} \\
&= \frac{(n-1)(n+5)+5}{5(n+4)(n+5)} \\
&= \frac{n^2+4n}{5(n+4)(n+5)} \\
&= \frac{n(n+4)}{5(n+4)(n+5)} \\
&= \frac{n}{5(n+5)} \\
&= \frac{(n+1)-1}{5(n+1)+20}.
\end{aligned}
$$

This completes the proof of the induction step.

Question 5 **(3 pts)**. Suppose A and B are subsets of a set X. Show that

$$(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}.$$

Solution: For $x \in X$, we have

$$
\begin{aligned}
x \in (A \cap B)^{\complement} &\iff \neg(x \in A \cap B) \\
&\iff \neg(x \in A \text{ and } x \in B) \\
&\iff x \notin A \text{ or } x \notin B \\
&\iff x \in A^{\complement} \text{ or } x \in B^{\complement} \\
&\iff x \in A^{\complement} \cup B^{\complement}.
\end{aligned}
$$

Question 6 **(4 pts)**. Consider the following subsets of $\mathbb{R} \times \mathbb{R}$:

$$A = \{(x, x+2) : x \in \mathbb{R}\}, \qquad B = \{(2y-1, 1+2y) : y \in \mathbb{R}\}.$$

Prove that $A = B$.

Solution: We first show that $A \subseteq B$. Let $(x, x+2)$, $x \in \mathbb{R}$, be an arbitrary element of A. If we define $y = \frac{x+1}{2} \in \mathbb{R}$, then $x = 2y - 1$, and so we have

$$(x, x+2) = (2y-1, 1+2y) \in B.$$

Hence $A \subseteq B$.

Now we show that $B \subseteq A$. Let $(2y-1, 1+2y)$, $y \in \mathbb{R}$, be an arbitrary element of B. If we define $x = 2y - 1$, then we have

$$(2y-1, 1+2y) = (x, x+2) \in A.$$

Hence $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, we have $A = B$.

QUESTION 7 **(5 pts)**. Let

$$X = \{x \in \mathbb{R} : x \neq 0\}$$

be the set of nonzero real numbers. Let $\sim$ be the relation on X defined by

$$x \sim y \iff xy^{-1} \in \mathbb{Z}.$$

(a) Is $\sim$ reflexive?

Solution: Yes, $\sim$ is reflexive. For all $x \in X$, we have $xx^{-1} = 1 \in \mathbb{Z}$, and so $x \sim x$.

(b) Is $\sim$ symmetric?

Solution: No, $\sim$ is not symmetric. For instance, $1, 2 \in X$ and $2 \sim 1$ since $2 \cdot 1^{-1} = 2 \in \mathbb{Z}$. However, $1 \cdot 2^{-1} = \frac{1}{2} \notin \mathbb{Z}$, and so $1 \nsim 2$.

(c) Is $\sim$ transitive?

Solution: Yes, $\sim$ is transitive. Suppose $x, y, z \in X$, $x \sim y$, and $y \sim z$. Thus

$$xy^{-1} \in \mathbb{Z} \quad \text{and} \quad yz^{-1} \in \mathbb{Z}.$$

Therefore,

$$xz^{-1} = \left(xy^{-1}\right)\left(yz^{-1}\right) \in \mathbb{Z},$$

since $\mathbb{Z}$ is closed under multiplication. Hence $x \sim z$.

(d) Is $\sim$ an equivalence relation?

Solution: No, $\sim$ is not an equivalence relation since it is not symmetric.

Remember to justify your answers.

QUESTION 8 **(4 pts)**.

(a) Fix $n \in \mathbb{N}$. State the definition of *congruence modulo* n. More precisely, complete the following sentence: "For $x, y \in \mathbb{Z}$, we say that x and y are congruent modulo n if and only if..."

Solution: For $x, y \in \mathbb{Z}$, we say that x and y are congruent modulo n if and only if $x - y$ is divisible by n.

(b) Suppose that $m, n \in \mathbb{N}$ and that m divides n. For $x, y \in \mathbb{Z}$, show that

$$(x \equiv y \mod n) \implies (x \equiv y \mod m).$$

Solution: Since m divides n, there exists $k \in \mathbb{Z}$ such that $n = km$. Suppose $x, y \in \mathbb{Z}$ satisfy $x \equiv y \mod n$. Then $x - y$ is divisible by n, and so there exists $a \in \mathbb{Z}$ such that

$$x - y = an = akm.$$

Thus $x - y$ is divisible by m, and so $x \equiv y \mod m$.

Question 9 **(7 pts)**. Consider the set

$$A = \left\{ 5 + \frac{2}{n} : n \in \mathbb{N} \right\}.$$

(a) Find the largest element (i.e. maximum) of A or show that it does not exist. In either case, remember to justify your answer.

Solution: We will show that $\max(A) = 7$. First note that

$$7 = 5 + \frac{2}{1} \in A.$$

Furthermore, we have

$$n \in \mathbb{N} \implies n \geq 1 \implies \frac{1}{n} \leq 1 \implies \frac{2}{n} \leq 2 \implies 5 + \frac{2}{n} \leq 7.$$

(b) Find the supremum of A or show that it does not exist. In either case, remember to justify your answer.

Solution: Since $\max(A) = 7$, we have $\sup(A) = 7$.

(c) Find the infimum of A or show that it does not exist. In either case, remember to justify your answer.

Solution: We will show that $\inf(A) = 5$. First note that

$$n \in \mathbb{N} \implies n > 0 \implies \frac{1}{n} > 0 \implies \frac{2}{n} > 0 \implies 5 + \frac{2}{n} > 5.$$

Hence 5 is a lower bound for A. It remains to show that it is the *greatest* lower bound. Suppose, towards a contradiction, that $b > 5$ is a lower bound for A. Then, $b - 5 \in \mathbb{R}_{>0}$ and, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} 5 + \frac{2}{n} > b &\implies \frac{2}{n} > b - 5 \\ &\implies \frac{n}{2} < \frac{1}{b-5} && \text{(since } b - 5 \in \mathbb{R}_{>0}\text{)} \\ &\implies n < \frac{2}{b-5}. \end{aligned}$$

This implies that $\frac{2}{b-5}$ is an upper bound for $\mathbb{N}$, contradicting the fact that $\mathbb{N}$ is not bounded above. It follows that $\inf(A) = 5$.

(d) Find the smallest element (i.e. minimum) of A or show that it does not exist. In either case, remember to justify your answer.

Solution: If A has a smallest element, then it is $\min(A) = \inf(A) = 5$. However, we showed above that $5 + \frac{2}{n} > 5$ for all $n \in \mathbb{N}$. Thus $5 \notin A$. So A does not have a smallest element.

QUESTION 10 **(4 pts)**. Suppose A and B are nonempty sets, and consider the function

$$f\colon A \times B \to A, \quad f(a,b) = a \quad \text{for } (a,b) \in A \times B.$$

(a) Is f surjective? Remember to justify your answer.

Solution: Yes, f is surjective. Since B is nonempty, we can choose $b \in B$. Then, for all $a \in A$, we have $f(a,b) = a$.

(b) When is f injective? More precisely, give conditions on the sets A and/or B that are satisfied if and only if f is injective.

Solution: The function f is injective if and only if B has exactly one element. Indeed, if $B = \{b\}$, then

$$f(a_1,b_1) = f(a_2,b_2) \implies a_1 = a_2 \implies (a_1,b_1) = (a_2,b_2),$$

where we use the fact that $b_1, b_2 \in B \implies b_1 = b = b_2$. On the other hand, if B has more than one element, then we can choose $b_1, b_2 \in B$ with $b_1 \neq b_2$. Then, choosing $a \in A$, we have

$$f(a,b_1) = a = f(a,b_2) \text{ but } (a,b_1) \neq (a,b_2).$$

QUESTION 11 **(4 pts)**.

(a) State the definition of a limit of a sequence of real numbers. In other words, if $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}$, state precisely what it means for the sequence to converge to $L \in \mathbb{R}$.

Solution: The sequence $(x_k)_{k=1}^{\infty}$ converges to L if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |x_n - L| < \varepsilon.$$

(b) What is

$$\lim_{n\to\infty} \left(4 + \frac{2n-1}{3n^2+5}\right)\ ?$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences, or the arithmetic of limits.

Solution: We will prove that the limit is 4. Note that, for $n \in \mathbb{N}$, we have

$$\left|\frac{2n-1}{3n^2+5}\right| = \frac{2n-1}{3n^2+5} \leq \frac{2n}{3n^2} = \frac{2}{3n}.$$

Let $\varepsilon > 0$. Since $\mathbb{N}$ has no upper bound, we can choose $N \in \mathbb{N}$ such that

$$N > \frac{2}{3\varepsilon}.$$

Then, for $n \geq N$, we have

$$\left|\left(4 + \frac{2n-1}{3n^2+5}\right) - 4\right| = \left|\frac{2n-1}{3n^2+5}\right| \leq \frac{2}{3n} \leq \frac{2}{3N} < \varepsilon.$$