University of Ottawa
Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs

Professor: Alistair Savage

Midterm Test (Version A) – Solutions

28 October 2019

Surname ________________________ First Name ______________________

Student # ______________________ DGD (1, 2, or 3) ____________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) Unless otherwise indicated, you must justify your answers to receive full marks.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.
(e) You may use the last page of the exam as scrap paper.

Cellular phones, unauthorized electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

By signing below, you acknowledge that you have read and ensured that you are complying with the above statement.

Signature: ____

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	3	6	5	4	5	4	27
Grade							

QUESTION 1. [**3 points**] Using *only the first five axioms of the integers* seen in class (as well as the replacement property), prove that

$$a \cdot 0 = 0 \quad \text{for all } a \in \mathbb{Z}.$$

Use only one axiom or definition per line of your proof, and state which axiom or definition you are using at each step (by its name).

Solution: Suppose $a \in \mathbb{Z}$. Then

$$\begin{aligned}
a \cdot (0+0) &= a \cdot 0 && \text{additive identity}\\
\implies a\cdot 0 + a \cdot 0 &= a \cdot 0 && \text{distributivity}\\
\implies (a\cdot 0 + a\cdot 0) + \big(-(a\cdot 0)\big) &= a\cdot 0 + \big(-(a\cdot 0)\big) && \text{replacement property}\\
\implies a\cdot 0 + \Big(a\cdot 0 + \big(-(a\cdot 0)\big)\Big) &= a\cdot 0 + \big(-(a\cdot 0)\big) && \text{associativity of addition}\\
\implies a\cdot 0 + 0 &= 0 && \text{additive inverse}\\
\implies a \cdot 0 &= 0. && \text{additive identity}
\end{aligned}$$

QUESTION 2. [**6 points**] In this question you do *not* need to justify your answers, except in part (d)(i).

(a) Complete the following sentence defining the ordering on the integers: "For $a, b \in \mathbb{Z}$, we write $a < b$ if and only if..."

Solution: For $a, b \in \mathbb{Z}$, we write $a < b$ if and only if $b - a \in \mathbb{N}$.

(b) Suppose $n \in \mathbb{N}$. Complete the following definition of *congruence modulo n*: "For $x, y \in \mathbb{Z}$, $x \equiv y$ if and only if..."

Solution: For $x, y \in \mathbb{Z}$,

$$x \equiv y \iff x - y \text{ is divisible by } n.$$

(c) State the well-ordering principle.

Solution: Every nonempty subset of $\mathbb{N}$ has a smallest element.

(d) Let P be the following statement:

$$n \geq 0 \implies n \in \mathbb{N}.$$

(Here we assume $n \in \mathbb{Z}$.)

(i) Is the statement P true or false? Explain your answer.

Solution: False, since $0 \geq 0$, but $0 \notin \mathbb{N}$.

(ii) Write the converse of the statement P. Is the converse true or false?

Solution: The converse is

$$n \in \mathbb{N} \implies n \geq 0.$$

This is true.

(iii) Write the contrapositive of the statement P. Is the contrapositive true or false?

Solution: The contrapositive is

$$n \notin \mathbb{N} \implies n < 0.$$

This is false.

QUESTION 3. Consider the sequence $(x_j)_{j=1}^{\infty}$ defined recursively by

$$x_1 = -5,$$
$$x_{n+1} = 3x_n + 20, \quad n \in \mathbb{N}.$$

(a) [**1 point**] What is x_4?

Solution: We have

$$x_2 = 3(-5) + 20 = 5$$
$$x_3 = 3 \cdot 5 + 20 = 35$$
$$x_4 = 3 \cdot 35 + 20 = 125.$$

(b) [**4 points**] Prove that x_n is divisible by 5 for all $n \in \mathbb{N}$.

Solution: *Base case*: When $n = 1$, we have

$$x_1 = -5 = 5(-1).$$

Thus the result holds for $n = 1$.

Induction step: Suppose that x_n is divisible by 5 for some $n \in \mathbb{N}$. Then there exists $m \in \mathbb{Z}$ such that $x_n = 5m$. Thus

$$\begin{aligned} x_{n+1} &= 3x_n + 20 \\ &= 3(5m) + 20 \\ &= 15m + 20 \\ &= 5(3m + 4). \end{aligned}$$

Since $m \in \mathbb{Z}$, we have that $3m + 4 \in \mathbb{Z}$. Hence x_{n+1} is divisible by 5, completing the proof of the induction step.

QUESTION 4. [**4 points**] Using induction, prove that

$$\sum_{k=6}^{n} \frac{1}{k^2 - 3k + 2} = \frac{n-5}{4n-4} \quad \text{for all } n \in \mathbb{Z},\ n \geq 6.$$

Solution: *Base case*: When $n = 6$, we have

$$\sum_{k=6}^{n} \frac{1}{k^2 - 3k + 2} = \frac{1}{6^2 - 3 \cdot 6 + 2} = \frac{1}{20} = \frac{6-5}{4 \cdot 6 - 4}.$$

Thus the result is true for $n = 6$.

Induction step: Suppose the result is true for some $n \in \mathbb{Z}$, $n \geq 6$. Then we have

$$\begin{aligned}
\sum_{k=6}^{n+1} \frac{1}{k^2 - 3k + 2} &= \sum_{k=6}^{n} \frac{1}{k^2 - 3k + 2} + \frac{1}{(n+1)^2 - 3(n+1) + 2} \\
&= \frac{n-5}{4n-4} + \frac{1}{n^2 - n} \\
&= \frac{n-5}{4(n-1)} + \frac{1}{n(n-1)} \\
&= \frac{n(n-5)+4}{4n(n-1)} \\
&= \frac{n^2 - 5n + 4}{4n(n-1)} \\
&= \frac{(n-4)(n-1)}{4n(n-1)} \\
&= \frac{n-4}{4n} \\
&= \frac{(n+1)-5}{4(n+1)-4}.
\end{aligned}$$

This proves the induction step.

QUESTION 5.

(a) [**2 points**] Suppose A and B are sets, and define

$$C = \{x : (x, x) \in A \times B\}.$$

Prove that

$$C = A \cap B.$$

Solution: We have

$$\begin{aligned} x \in C &\iff (x, x) \in A \times B \\ &\iff x \in A \text{ and } x \in B \\ &\iff x \in A \cap B. \end{aligned}$$

(b) [**3 points**] Suppose $A, B \subseteq X$. Prove that

$$(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}.$$

Solution: For $x \in X$, we have

$$\begin{aligned} x \in (A \cup B)^{\complement} &\iff \neg\big(x \in A \cup B\big) \\ &\iff \neg\big(x \in A \text{ or } x \in B\big) \\ &\iff x \notin A \text{ and } x \notin B \\ &\iff x \in A^{\complement} \text{ and } x \in B^{\complement} \\ &\iff x \in A^{\complement} \cap B^{\complement}. \end{aligned}$$

QUESTION 6. [**4 points**] Define a relation $\sim$ on the set $\mathbb{N} \times \mathbb{Z}$ by

$$(a, b) \sim (c, d) \iff 3(a - c) + 5(d - b) = 0.$$

Prove that $\sim$ is an equivalence relation.

Solution: *Reflexivity*: Suppose $(a, b) \in \mathbb{N} \times \mathbb{Z}$. Since

$$3(a - a) + 5(b - b) = 0$$

we have $(a, b) \sim (a, b)$.

Symmetry: Suppose $(a, b), (c, d) \in \mathbb{N} \times \mathbb{Z}$ such that $(a, b) \sim (c, d)$. Then

$$\begin{aligned} (a, b) \sim (c, d) &\implies 3(a - c) + 5(d - b) = 0 \\ &\implies (-1)\big(3(a - c) + 5(d - b)\big) = 0 \\ &\implies 3(c - a) + 5(b - d) = 0 \\ &\implies (c, d) \sim (a, b). \end{aligned}$$

Transitivity: Suppose $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{Z}$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. So we have

$$3(a - c) + 5(d - b) = 0 \qquad \text{and} \qquad 3(c - e) + 5(f - d) = 0.$$

Adding these two equations together gives

$$3(a - c) + 5(d - b) + 3(c - e) + 5(f - d) = 0.$$

Simplifying then gives

$$3(a - e) + 5(f - b) = 0,$$

which implies that $(a, b) \sim (e, f)$.