University of Ottawa
Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs
Professor: Alistair Savage

Final Exam
December 2019

Surname ______________________ First Name ______________________

Student # ______________________ Seat # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) Unless otherwise indicated, you must justify your answers to receive full marks.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.

Cellular phones, unauthorized electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

By signing below, you acknowledge that you have read and ensured that you are complying with the above statement.

Signature: ____

Question	1	2	3	4	5	6	7	8	9	10	11	12	Total
Maximum	3	6	4	4	3	3	5	3	5	4	7	4	51
Grade													

QUESTION 1. [**3 points**] Suppose $a, b, c, d \in \mathbb{Z}$. Prove that

$$a(b + cd) = c(ad) + ab.$$

You should *only* use the first five *axioms* for the integers we saw in class. Do *not* use any propositions. Use only one axiom at each step, and clearly indicate, by name, which axiom you are using at each step.

Question 2. [**6 points**] In this question, you do *not* need to justify your answers.

(a) State the additive identity axiom for the integers.

(b) State the completeness axiom for the real numbers.

(c) State the binomial theorem for integers.

(d) State Fermat's Little Theorem.

(e) Give an example of a relation on $\mathbb{R}$ that is reflexive and transitive, but not symmetric.

(f) For which $a \in \mathbb{N}$ is $\sqrt{a}$ irrational?

QUESTION 3. [**4 points**] Suppose $a, b \in \mathbb{Z}$. Let P be the following statement:

If 3 divides a and 8 divides b, then 6 divides ab.

(a) Prove that the statement P is true.

(b) Write the converse of the statement P. Is the converse of P true or false? Remember to justify your answer.

(c) Write the contrapositive of the statement P. Is the contrapositive of P true or false?

QUESTION 4. [**4 points**] Determine the negation of the following statements.

(a) $(\exists x, y \in \mathbb{Z} \text{ such that})\, x^2 = y^2$ and $x \neq y$.

(b) For every real number b, we have $b^3 \geq 0$ or $b = 0$.

(c) If the Earth is flat, then I will eat my hat.

(d) $(\forall \varepsilon > 0)(\exists \delta > 0 \text{ such that})(\forall x, y \in \mathbb{R})\big(|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon\big)$

QUESTION 5. [**3 points**] Suppose A, B, C, and D are sets. Prove that

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D).$$

Make sure you are using appropriate mathematical notation throughout.

Question 6. [**3 points**] Consider the sets

$$S = \{2n + 1 : n \in \mathbb{Z}\} \quad \text{and} \quad T = \{6m + 7 : m \in \mathbb{Z}\}.$$

(a) Is S a subset of T? If so, prove it; otherwise, justify why $S \not\subseteq T$.

(b) Is T a subset of S? If so, prove it; otherwise, justify why $T \not\subseteq S$.

Question 7. [**5 points**]

(a) Determine the sum of the finite series $\sum_{j=1}^{3} \frac{1}{j^2+j}$. Remember to show your work.

(b) Using induction, prove that $\sum_{j=1}^{n} \frac{1}{j^2+j} = \frac{n}{n+1}$ for all integers $n \geq 1$.

Question 8. **[3 points]**

(a) Perform the following computations in $\mathbb{Z}_5$ and write your final answer in the standard form: [0], [1], [2], [3], or [4].

(i) $[7] \odot ([3] \oplus [6])$

(ii) $([-3] \odot [24]) \oplus [6667] \oplus [50]$

(b) Does [3] have a multiplicative inverse in $\mathbb{Z}_8$? If so, what is it? Remember to justify your answer.

QUESTION 9.

(a) [**5 points**] Define the relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ by

$$(a, b) \sim (c, d) \iff ad = bc.$$

where $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Prove that $\sim$ is an equivalence relation.

(b) Does $(-5, -5)$ belong to the equivalence class of $(2, 2)$ with respect to $\sim$? Justify your answer.

QUESTION 10. [**4 points**] Consider the set

$$A = \left\{ \frac{-3}{x} + 7 : x \in \mathbb{R}_{>0} \right\}.$$

(a) Find the supremum of A or justify that it does not exist.

(b) Find the largest element (i.e. maximum) of A or justify that it does not exist.

Question 11. [**7 points**] Consider the function

$$f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad f(x) = 5x + 2.$$

(a) Is f injective? Justify your answer.

(b) Is f surjective? Justify your answer.

(c) Does f have a left inverse? Justify your answer.

(d) Does f have a right inverse? Justify your answer.

(e) Does f have a two-sided inverse? If it does, give one and show that it is indeed a two-sided inverse. Otherwise, justify why f does not have a two-sided inverse.

QUESTION 12. [**4 points**] Find the limit of the sequence

$$\left(8 - \frac{2}{n^2 - 3}\right)_{n=2}^{\infty}.$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences or the arithmetic of limits.