University of Ottawa

Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs

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Final Exam – Solutions

December 2019

Surname ____________________ First Name ____________________

Student # ____________________ Seat # ____________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) Unless otherwise indicated, you must justify your answers to receive full marks.
(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(d) Write your student number at the top of each page in the space provided.

Cellular phones, unauthorized electronic devices or course notes are not allowed during this test. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession such as in your pockets. If caught with such a device or document, academic fraud allegations may be filed which may result in you obtaining zero for this test.

By signing below, you acknowledge that you have read and ensured that you are complying with the above statement.

Signature: ____

Question	1	2	3	4	5	6	7	8	9	10	11	12	Total
Maximum	3	6	4	4	3	3	5	3	5	4	7	4	51
Grade													

QUESTION 1. [**3 points**] Suppose $a, b, c, d \in \mathbb{Z}$. Prove that

$$a(b + cd) = c(ad) + ab.$$

You should *only* use the first five *axioms* for the integers we saw in class. Do *not* use any propositions. Use only one axiom at each step, and clearly indicate, by name, which axiom you are using at each step.

Solution: We have

$$\begin{aligned}
a(b + cd) &= ab + a(cd) && \text{(distributivity)}\\
&= a(cd) + ab && \text{(commutativity of addition)}\\
&= (cd)a + ab && \text{(commutativity of multiplication)}\\
&= c(da) + ab && \text{(associativity of multiplication)}\\
&= c(ad) + ab. && \text{(commutativity of multiplication)}
\end{aligned}$$

QUESTION 2. [**6 points**] In this question, you do *not* need to justify your answers.

(a) State the additive identity axiom for the integers.

Solution: There exists an integer 0 such that $a + 0 = a$ for all $a \in \mathbb{Z}$.

(b) State the completeness axiom for the real numbers.

Solution: Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

(c) State the binomial theorem for integers.

Solution: If $a, b \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, then

$$(a+b)^k = \sum_{m=0}^{k} \binom{k}{m} a^m b^{k-m}.$$

(d) State Fermat's Little Theorem.

Solution: If $m \in \mathbb{Z}$ and p is prime, then

$$m^p \equiv m \mod p.$$

(e) Give an example of a relation on $\mathbb{R}$ that is reflexive and transitive, but not symmetric.

Solution: One example is the relation $\leq$ of "less than or equal to".

(f) For which $a \in \mathbb{N}$ is $\sqrt{a}$ irrational?

Solution: The real number $\sqrt{a}$ is irrational if and only if a is not a perfect square.

QUESTION 3. [**4 points**] Suppose $a, b \in \mathbb{Z}$. Let P be the following statement:

If 3 divides a and 8 divides b, then 6 divides ab.

(a) Prove that the statement P is true.

Solution: Suppose that 3 divides a and 8 divides b. Then $a = 3m$ and $b = 8n$ for some $m, n \in \mathbb{Z}$. Thus

$$ab = (3m)(8n) = 24(mn) = 6 \cdot (4mn).$$

(b) Write the converse of the statement P. Is the converse of P true or false? Remember to justify your answer.

Solution: The converse is

If 6 divides ab, then 3 divides a and 8 divides b.

This is false. For example, if $a = 1$ and $b = 6$, we have that 6 divides ab but 3 does not divide a and 8 does not divide b.

(c) Write the contrapositive of the statement P. Is the contrapositive of P true or false?

Solution: The contrapositive is:

If 6 does not divide ab, then 3 does not divide a or 8 does not divide b.

This is true, since P is true, and a statement is equivalent to its contrapositive.

QUESTION 4. [**4 points**] Determine the negation of the following statements.

(a) $(\exists x, y \in \mathbb{Z} \text{ such that})\, x^2 = y^2$ and $x \neq y$.

Solution: $(\forall x, y \in \mathbb{Z})\, x^2 \neq y^2$ or $x = y$

(b) For every real number b, we have $b^3 \geq 0$ or $b = 0$.

Solution: There is a real number b such that $b^3 < 0$ and $b \neq 0$.

(c) If the Earth is flat, then I will eat my hat.

Solution: The Earth is flat and I will *not* eat my hat.

(d) $(\forall \varepsilon > 0)(\exists \delta > 0 \text{ such that})(\forall x, y \in \mathbb{R})\big(|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon\big)$

Solution:

$(\exists \varepsilon > 0 \text{ such that})(\forall \delta > 0)(\exists x, y \in \mathbb{R} \text{ such that})\big(|x - y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon\big)$

QUESTION 5. [**3 points**] Suppose A, B, C, and D are sets. Prove that

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D).$$

Make sure you are using appropriate mathematical notation throughout.

Solution: Let A, B, C, D be sets. Then

$$\begin{aligned}
(x, y) \in (A \cap B) \times (C \cap D) &\iff x \in A \cap B \text{ and } y \in C \cap D \\
&\iff x \in A \text{ and } x \in B \text{ and } y \in C \text{ and } y \in D \\
&\iff x \in A \text{ and } y \in C \text{ and } x \in B \text{ and } y \in D \\
&\iff (x, y) \in A \times C \text{ and } (x, y) \in B \times D \\
&\iff (x, y) \in (A \times C) \cap (B \times D).
\end{aligned}$$

Therefore, $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

QUESTION 6. [**3 points**] Consider the sets

$$S = \{2n+1 : n \in \mathbb{Z}\} \quad \text{and} \quad T = \{6m+7 : m \in \mathbb{Z}\}.$$

(a) Is S a subset of T? If so, prove it; otherwise, justify why $S \not\subseteq T$.

Solution: We have $S \not\subseteq T$ because there exists an element of S that is not an element of T. For example, we have $3 \in S$, but $3 \notin T$ since there is no integer $m \in \mathbb{Z}$ such that $3 = 6m + 7$.

(b) Is T a subset of S? If so, prove it; otherwise, justify why $T \not\subseteq S$.

Solution: We have $T \subseteq S$ as follows: Assume $x \in T$. Then $x = 6m + 7$ for some integer $m \in \mathbb{Z}$. Thus, $x = 2(3m+3)+1$. Since $m \in \mathbb{Z}$, we see that $3m+3 \in \mathbb{Z}$. Thus $x = 2n + 1$ for $n = 3m + 3$, and so $x \in S$.

Question 7. **[5 points]**

(a) Determine the sum of the finite series $\sum_{j=1}^{3} \frac{1}{j^2+j}$. Remember to show your work.

Solution: We have

$$\sum_{j=1}^{3} \frac{1}{j^2+j} = \frac{1}{1^2+1} + \frac{1}{2^2+2} + \frac{1}{3^2+3} = \frac{6}{12} + \frac{2}{12} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}.$$

(b) Using induction, prove that $\sum_{j=1}^{n} \frac{1}{j^2+j} = \frac{n}{n+1}$ for all integers $n \geq 1$.

Solution: *Base case:* When $n = 1$, we have

$$\sum_{j=1}^{n} \frac{1}{j^2+j} = \sum_{j=1}^{1} \frac{1}{j^2+j} = \frac{1}{1^2+1} = \frac{1}{2} \quad \text{and} \quad \frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}.$$

Thus, the result is true for $n = 1$.

Induction step: Assume

$$\sum_{j=1}^{n} \frac{1}{j^2+j} = \frac{n}{n+1}$$

holds true for *some* integer $n \geq 1$.

Then

$$\begin{aligned}
\sum_{j=1}^{n+1} \frac{1}{j^2+j} &= \sum_{j=1}^{n} \frac{1}{j^2+j} + \frac{1}{(n+1)^2+(n+1)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)^2+(n+1)} && \text{(by the induction hypothesis)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{(n+1)}{(n+2)} \\
&= \frac{(n+1)}{(n+1)+1}.
\end{aligned}$$

Thus, the result holds for $n+1$, which completes the proof of the induction step.

QUESTION 8. **[3 points]**

(a) Perform the following computations in $\mathbb{Z}_5$ and write your final answer in the standard form: [0], [1], [2], [3], or [4].

(i) $[7] \odot ([3] \oplus [6])$

Solution: $[7] \odot ([3] \oplus [6]) = [2] \odot ([3] \oplus [1]) = [2] \odot [4] = [8] = [3]$.

(ii) $([-3] \odot [24]) \oplus [6667] \oplus [50]$

Solution: $([-3] \odot [24]) \oplus [6667] \oplus [50] = ([2] \odot [4]) \oplus [2] \oplus [0] = [10] = [0]$.

(b) Does [3] have a multiplicative inverse in $\mathbb{Z}_8$? If so, what is it? Remember to justify your answer.

Solution: Yes, it does. Since

$$[3] \odot [3] = [9] = [1],$$

the element [3] is its own multiplicative inverse.

Question 9.

(a) [**5 points**] Define the relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ by

$$(a, b) \sim (c, d) \iff ad = bc.$$

where $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Prove that $\sim$ is an equivalence relation.

Solution: *Reflexivity*: Suppose $(a, b) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Since $ab = ba$, we have $(a, b) \sim (a, b)$. So $\sim$ is reflexive.

Symmetry: Suppose $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ such that $(a, b) \sim (c, d)$. Then $ad = bc$. Thus $cb = da$, and so $(c, d) \sim (a, b)$. Hence $\sim$ is symmetric.

Transitivity: Suppose $(a, b), (c, d), (f, g) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (f, g)$. Then $ad = bc$ and $cg = df$. So

$$adg = bcg = bdf$$

Since $d \neq 0$, the cancellation property gives $ag = bf$. Hence $(a, b) \sim (f, g)$. So $\sim$ is transitive.

(b) Does $(-5, -5)$ belong to the equivalence class of $(2, 2)$ with respect to $\sim$? Justify your answer.

Solution: Yes, it does, since $(-5) \cdot 2 = (-5) \cdot 2$. Thus $(-5, -5) \sim (2, 2)$.

QUESTION 10. [**4 points**] Consider the set

$$A = \left\{ \frac{-3}{x} + 7 : x \in \mathbb{R}_{>0} \right\}.$$

(a) Find the supremum of A or justify that it does not exist.

Solution: We will show that $\sup(A) = 7$. First note that

$$\begin{aligned} x \in \mathbb{R}_{>0} &\implies x > 0 \\ &\implies \frac{1}{x} > 0 \\ &\implies \frac{-3}{x} < 0 \\ &\implies \frac{-3}{x} + 7 < 7. \end{aligned}$$

So 7 is an upper bound for A. It remains to show that it is the *least* upper bound, which we do by contradiction. Suppose y is an upper bound for A with $y < 7$ (so $7 - y > 0$). We will arrive at a contradiction by finding $x \in \mathbb{R}_{>0}$ such that $\frac{-3}{x} + 7 > y$. Note that, for $x \in \mathbb{R}_{>0}$, we have

$$\begin{aligned} \frac{-3}{x} + 7 > y &\iff 7 - y > \frac{3}{x} \\ &\iff \frac{1}{7-y} < \frac{x}{3} \qquad \text{(since } x, 3, 7 - y > 0\text{)} \\ &\iff \frac{3}{7-y} < x. \end{aligned}$$

Now, $\mathbb{R}_{>0}$ is not bounded above, there exist $x \in \mathbb{R}_{>0}$ such that $\frac{3}{7-y} < x$. Then the above shows that $\frac{-3}{x} + 7 > y$, contradicting the assumption that y is an upper bound for A.

(b) Find the largest element (i.e. maximum) of A or justify that it does not exist.

Solution: If A had a largest element, we would have $\max(A) = \sup(A) = 7$. However, as shown above, all elements of A are *strictly* less that 7. So $7 \notin A$. Hence A does not have a largest element.

QUESTION 11. [**7 points**] Consider the function

$$f: \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad f(x) = 5x + 2.$$

(a) Is f injective? Justify your answer.

Solution: Yes, f is injective. Suppose $x, y \in \mathbb{R}_{\geq 0}$. Then

$$\begin{aligned} f(x) = f(y) &\implies 5x + 2 = 5y + 2 \\ &\implies 5x = 5y \\ &\implies x = y. \end{aligned}$$

(b) Is f surjective? Justify your answer.

Solution: No, f is not surjective. Since

$$x \in \mathbb{R}_{\geq 0} \implies x \geq 0 \implies 5x \geq 0 \implies f(x) = 5x + 2 \geq 2,$$

we can see, for instance, that $0 \in \mathbb{R}$ is not in the image of f.

(c) Does f have a left inverse? Justify your answer.

Solution: Yes. A function has a left inverse if and only if it is injective. Since f is injective, it has a left inverse.

(d) Does f have a right inverse? Justify your answer.

Solution: No. A function has a right inverse if and only if it is surjective. Since f is not surjective, it has no right inverse.

(e) Does f have a two-sided inverse? If it does, give one and show that it is indeed a two-sided inverse. Otherwise, justify why f does not have a two-sided inverse.

Solution: No. Since f does not have a right inverse, it does not have a two-sided inverse.

QUESTION 12. [**4 points**] Find the limit of the sequence

$$\left(8 - \frac{2}{n^2 - 3}\right)_{n=2}^{\infty}.$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences or the arithmetic of limits.

Solution: We will show that

$$\lim_{n\to\infty}\left(8 - \frac{2}{n^2 - 3}\right) = 8.$$

Let $\varepsilon > 0$. Note that

$$\left|8 - \frac{2}{n^2 - 3} - 8\right| = \left|\frac{-2}{n^2 - 3}\right| = \frac{2}{n^2 - 3} \quad (\text{since } n \geq 2 \text{ implies } n^2 - 3 > 0).$$

Since $\mathbb{N}$ is unbounded, we can choose $N \in \mathbb{N}$ such that $N \geq \sqrt{\frac{2}{\varepsilon} + 3}$. Then, for all $n \geq N$, we have

$$\left|8 - \frac{2}{n^2 - 3} - 8\right| = \frac{2}{n^2 - 3} \leq \frac{2}{N^2 - 3} \leq \frac{2}{\left(\sqrt{\frac{2}{\varepsilon} + 3}\right)^2 - 3} = \varepsilon.$$