University of Ottawa

Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs

Professor: Alistair Savage

Midterm Test – Solutions

1 November 2018

Surname ______________________ First Name ______________________

Student # ______________________ DGD (1, 2, or 3) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.

(b) Unless otherwise indicated, you must justify your answers to receive full marks.

(c) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.

(d) Write your student number at the top of each page in the space provided.

(e) No notes, books, scrap paper, calculators or other electronic devices are allowed.

(f) You should write in *pen*, not pencil.

(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	4	4	2	3	6	4	23
Grade							

QUESTION 1. [**4 points**] Suppose $a, b, x \in \mathbb{Z}$ such that $a \neq 0$ and $a(x+b) = a$. Using *only the first five axioms of the integers* seen in class (as well as the replacement property and the definition of subtraction), prove that $x = 1 - b$.

Use only one axiom or definition per line of your proof, and state which axiom or definition you are using at each step (by its name or number).

Solution: We have

$$
\begin{aligned}
& a(x+b) = a \\
\implies\ & a(x+b) = a \cdot 1 && \text{multiplicative identity} \\
\implies\ & x + b = 1 && \text{cancellation} \\
\implies\ & (x+b) + (-b) = 1 + (-b) && \text{replacement} \\
\implies\ & x + (b + (-b)) = 1 + (-b) && \text{associativity of addition} \\
\implies\ & x + 0 = 1 + (-b) && \text{additive inverse} \\
\implies\ & x = 1 + (-b) && \text{additive identity} \\
\implies\ & x = 1 - b. && \text{definition of subtraction}
\end{aligned}
$$

QUESTION 2. [**4 points**] Prove by induction that

$$\sum_{k=3}^{n} \frac{2}{k^2-3k+2} = \frac{2n-4}{n-1} \quad \text{for all } n \in \mathbb{Z},\ n \geq 3.$$

Solution: *Base case*: When $n = 3$, we have

$$\sum_{k=3}^{3} \frac{2}{k^2-3k+2} = \frac{2}{3^2-3\cdot 3+2} = \frac{2}{2} = \frac{2\cdot 3-4}{3-1}.$$

Thus the result holds for $n = 3$.

Induction step: Suppose that, for some $n \in \mathbb{Z}$, $n \geq 3$, we have

$$\sum_{k=3}^{n} \frac{2}{k^2-3k+2} = \frac{2n-4}{n-1}.$$

Then

$$\begin{aligned}\sum_{k=3}^{n+1} \frac{2}{k^2-3k+2} &= \sum_{k=3}^{n} \frac{2}{k^2-3k+2} + \frac{2}{(n+1)^2-3(n+1)+2} \\ &= \frac{2n-4}{n-1} + \frac{2}{n^2-n} \\ &= \frac{2n-4}{n-1} + \frac{2}{n(n-1)} \\ &= \frac{2n^2-4n+2}{n(n-1)} \\ &= \frac{2(n-1)^2}{n(n-1)} \\ &= \frac{2(n-1)}{n} \\ &= \frac{2(n+1)-4}{(n+1)-1}.\end{aligned}$$

This completes the proof of the induction step.

QUESTION 3. [**2 points**] For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. Your grade on this question will then be rounded down to the nearest 0.5 points. You cannot receive a negative score on this question.

T $(\exists a \in \mathbb{N} \text{ such that})(\forall b \in \mathbb{Z})\, a < b^2 + 2$.

T $(\forall a \in \mathbb{N})\,(\exists b \in \mathbb{Z} \text{ such that})\, ab = 0$.

F $(\forall x \in \mathbb{Z})\,(\exists y \in \mathbb{N} \text{ such that})\, y > y + x$.

T $(\exists x \in \mathbb{Z} \text{ such that})(\forall y \in \mathbb{N})\, y > y + x$.

QUESTION 4. [**3 points**] Consider the following statement:

$$(0 \cdot 2 = 1) \implies (0 \leq -1)$$

Give the converse and the contrapositive of this statement. Is the original statement true? Is the converse true? Is the contrapositive true? Justify your answers.

Solution: The original statement is true since the hypothesis is false. (We have $0 \cdot 2 = 0 \neq 1$.)

Converse: $(0 \leq -1) \implies (0 \cdot 2 = 1)$. This statement is true since the hypothesis is false.

Contrapositive: $(0 > -1) \implies (0 \cdot 2 \neq 1)$. This statement is true since both the hypothesis and the conclusion are true. (Also, since any statement is equivalent to its contrapositive, this contrapositive is true because the original statement is true.)

QUESTION 5.

(a) **[3 points]** Suppose $A, B \subseteq X$. Prove that
$$(A - B)^{\complement} = A^{\complement} \cup B.$$

Solution: For $x \in X$, we have
$$\begin{aligned} x \in (A-B)^{\complement} &\iff \neg(x \in A - B) \\ &\iff \neg(x \in A \text{ and } x \notin B) \\ &\iff x \notin A \text{ or } x \in B \\ &\iff x \in A^{\complement} \text{ or } x \in B \\ &\iff x \in A^{\complement} \cup B. \end{aligned}$$
Thus $(A - B)^{\complement} = A^{\complement} \cup B$.

(b) **[3 points]** Consider the following subsets of $\mathbb{Z} \times \mathbb{Z}$:
$$A = \{(3n, 3m+1) : n, m \in \mathbb{Z}\} \quad \text{and} \quad B = \{(k, k^2) : k \in \mathbb{Z}\}.$$
Prove that $A \cap B = \varnothing$.

Solution: Suppose, towards a contradiction that there is some element $(x, y) \in A \cap B$. Since $(x, y) \in A$, there exists some $n, m \in \mathbb{Z}$ such that
$$x = 3n \quad \text{and} \quad y = 3m + 1.$$
Similarly, since $(x, y) \in B$, there exists some $k \in \mathbb{Z}$ such that
$$x = k \quad \text{and} \quad y = k^2.$$
Thus, we have
$$3m + 1 = k^2 = (3n)^2.$$
This implies that
$$1 = (3n)^2 - 3m = 9n^2 - 3m = 3(n^2 - m),$$
and so 1 is divisible by 3, which is a contradiction.

Since our assumption that there is an element in $A \cap B$ lead to a contradiction, we must have $A \cap B = \varnothing$.

QUESTION 6. [**4 points**] Prove that

$$(a, b) \sim (c, d) \iff 2(b - d) = 3(a - c)$$

defines an equivalence relation on $\mathbb{Z} \times \mathbb{Z}$.

Solution: *Reflexivity*: Suppose $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Since

$$2(b - b) = 0 = 3(a - a),$$

we have $(a, b) \sim (a, b)$.

Symmetry: Suppose $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$ such that $(a, b) \sim (c, d)$. Then

$$\begin{aligned} (a, b) \sim (c, d) &\implies 2(b - d) = 3(a - c) \\ &\implies -2(b - d) = -3(a - c) \\ &\implies 2(d - b) = 3(c - a) \\ &\implies (c, d) \sim (a, b). \end{aligned}$$

Transitivity: Suppose $(a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{Z}$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. So we have

$$2(b - d) = 3(a - c) \qquad \text{and} \qquad 2(d - f) = 3(c - e).$$

Adding these two equations together gives

$$2(b - d) + 2(d - f) = 3(a - c) + 3(c - e).$$

Simplifying then gives

$$2(b - f) = 3(a - e),$$

which implies that $(a, b) \sim (e, f)$.