University of Ottawa

Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs

Professor: Alistair Savage

Final Exam

December 2018

Surname ______________________ First Name ______________________

Student # ______________________ Seat # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	Total
Maximum	4	4	3	3	4	4	5	7	6	4	44
Grade											

QUESTION 1. [**4 points**] For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer. (You will not lose points for incorrect answers.)

____ $(\forall x \in \mathbb{R})\ (\exists n \in \mathbb{Z} \text{ such that})\ n < x$.

____ $(\forall x \in (-\infty, 0))\ (\exists y \in \mathbb{R} \text{ such that})\ x < y < 0$

____ $(\forall x \in \mathbb{R})\ (\exists!\, n \in \mathbb{Z} \text{ such that})\ n \leq x < n + 1$

____ $(\exists a \in \mathbb{Z} \text{ such that})\ a(-a) \in \mathbb{N}$

____ $1 = 0 \iff (\forall n \in \mathbb{Z}, n = 0)$

____ $0 \leq 1 \iff 0 = 1$

____ Every bounded sequence of real numbers converges.

____ If $A, B \subseteq \mathbb{R}$ and $\sup(A) \leq \sup(B)$, then $A \subseteq B$.

QUESTION 2. [**4 points**] Give the negation of each of the following statements. Simplify your answer as much as possible.

(a) $P \iff Q$. (In your answer, the symbol $\neg$ should only appear directly in front of P or Q.)

(b) $\forall n \in \mathbb{Z}$, n, $n+1$, or $n+2$ is divisible by 3.

(c) Every function $f\colon A \to B$ has the property that $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

(d) $\forall x \in \mathbb{R}$, $(x^2 > 1 \implies x > 1)$.

QUESTION 3. [**3 points**] Prove that, for all sets A and B,

$$A \cup B = B \iff A \subseteq B.$$

QUESTION 4. [**3 points**] Let

$$A = \{(n, 2-n) : n \in \mathbb{Z}\} \qquad \text{and} \qquad B = \{(x, x^2) : x \in \mathbb{R}\}.$$

Find all elements of $A \cap B$.

QUESTION 5. [**4 points**] Prove by induction that

$$2n^2 > 5n + 5 \quad \text{for all } n \in \mathbb{N}, n \geq 4.$$

QUESTION 6. [**4 points**] Use modular arithmetic to show that, for all $n \in \mathbb{Z}$,

$$n^5 + 8n^4 + 9n^3 - 26n^2 - 52n - 24$$

is divisible by 3.

QUESTION 7. [**5 points**] Let $\sim$ be the relation on $\mathbb{R}_{>0}$ defined by

$$r \sim s \iff rs^{-1} \in \mathbb{Q}.$$

Prove that $\sim$ is an equivalence relation.

QUESTION 8. [**7 points**] Consider the set

$$A = \left\{ \frac{2}{n} - 6 : n \in \mathbb{N} \right\}.$$

(a) Find the largest element (i.e. maximum) of A or prove that it does not exist.

(b) Find the supremum of A or prove that it does not exist.

(c) Find the infimum of A or prove that it does not exist.

(d) Find the smallest element (i.e. minimum) of A or prove that it does not exist.

QUESTION 9. [**6 points**] Consider the function

$$f\colon \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad f(x) = (x-1)^2.$$

(a) Is f injective? Justify your answer.

(b) Is f surjective? Justify your answer.

(c) Does f have a left inverse? If it does, give one and show that it is indeed a left inverse. Otherwise, justify why f does not have a left inverse.

(d) Does f have a right inverse? If it does, give one and show that it is indeed a right inverse. Otherwise, justify why f does not have a right inverse.

QUESTION 10. [**4 points**] Find the limit of the sequence

$$\left(\frac{5}{2} + \frac{1}{n^2+1}\right)_{n=1}^{\infty}.$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences or the arithmetic of limits.

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