University of Ottawa

Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs

Professor: Alistair Savage

Final Exam – Solutions

December 2018

Surname ______________________ First Name ______________________

Student # ______________________ Seat # ______________________

Instructions:

(a) You have 3 hours to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	8	9	10	Total
Maximum	4	4	3	3	4	4	5	7	6	4	44
Grade											

QUESTION 1. [**4 points**] For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer. (You will not lose points for incorrect answers.)

T $(\forall x \in \mathbb{R})$ $(\exists n \in \mathbb{Z}$ such that$)$ $n < x$.

T $(\forall x \in (-\infty, 0))$ $(\exists y \in \mathbb{R}$ such that$)$ $x < y < 0$

T $(\forall x \in \mathbb{R})$ $(\exists!\, n \in \mathbb{Z}$ such that$)$ $n \leq x < n+1$

F $(\exists a \in \mathbb{Z}$ such that$)$ $a(-a) \in \mathbb{N}$

T $1 = 0 \iff (\forall n \in \mathbb{Z}, n = 0)$

F $0 \leq 1 \iff 0 = 1$

F Every bounded sequence of real numbers converges.

F If $A, B \subseteq \mathbb{R}$ and $\sup(A) \leq \sup(B)$, then $A \subseteq B$.

QUESTION 2. [**4 points**] Give the negation of each of the following statements. Simplify your answer as much as possible.

(a) $P \iff Q$. (In your answer, the symbol $\neg$ should only appear directly in front of P or Q.)

Solution: $(\neg P$ and $Q)$ or $(P$ and $\neg Q)$.

(b) $\forall n \in \mathbb{Z}$, n, $n+1$, or $n+2$ is divisible by 3.

Solution: $\exists n \in \mathbb{Z}$ such that $3 \nmid n$ and $3 \nmid (n+1)$ and $3 \nmid (n+2)$.

(c) Every function $f\colon A \to B$ has the property that $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$.

Solution: There exists a function $f\colon A \to B$ for which $\exists b \in B$ such that $\forall a \in A$, $f(a) \neq b$.

(d) $\forall x \in \mathbb{R}$, $(x^2 > 1 \implies x > 1)$.

Solution: $\exists x \in \mathbb{R}$ such that $x^2 > 1$ and $x \leq 1$.

QUESTION 3. [**3 points**] Prove that, for all sets A and B,

$$A \cup B = B \iff A \subseteq B.$$

Solution: First suppose $A \subseteq B$. We wish to show that $A \cup B = B$. Since we always have $B \subseteq A \cup B$, it suffices to prove the reverse inclusion. We have

$$\begin{aligned} x \in A \cup B &\implies x \in A \text{ or } x \in B \\ &\implies x \in B \text{ or } x \in B && \text{(since } A \subseteq B\text{)} \\ &\implies x \in B. \end{aligned}$$

So $A \cup B \subseteq B$ as desired.

Now suppose $A \cup B = B$. Then

$$\begin{aligned} x \in A &\implies x \in A \cup B && \text{(since } A \subseteq A \cup B\text{)} \\ &\implies x \in B. && \text{(since } A \cup B = B\text{)} \end{aligned}$$

So $A \subseteq B$.

Alternative proof of second part: Suppose $A \not\subseteq B$. Then there exists $x \in A$ such that $x \notin B$. Thus $x \in A \cup B$ but $x \notin B$. So $A \cup B \neq B$.

QUESTION 4. [**3 points**] Let

$$A = \{(n, 2-n) : n \in \mathbb{Z}\} \qquad \text{and} \qquad B = \{(x, x^2) : x \in \mathbb{R}\}.$$

Find all elements of $A \cap B$.

Solution: An element y is in $A \cap B$ if and only if there exist $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ such that

$$(n, 2-n) = y = (x, x^2).$$

Such n and x exist if and only if $n = x$ and $x^2 = (2-n)$. Now

$$(2-n) = n^2 \iff n^2 + n - 2 = 0 \iff (n+2)(n-1) = 0 \iff n = 1 \text{ or } n = -2.$$

Thus

$$A \cap B = \{(1,1), (-2,4)\}.$$

QUESTION 5. [**4 points**] Prove by induction that

$$2n^2 > 5n + 5 \quad \text{for all } n \in \mathbb{N}, n \geq 4.$$

Solution: We prove the result by induction on n.

Base case: When $n = 4$, we have

$$2 \cdot 4^2 = 32 > 25 = 5 \cdot 4 + 5.$$

Thus the result holds when $n = 4$.

Induction step: Suppose the result holds for some $n \geq 4$. Then

$$\begin{aligned} 2(n+1)^2 &= 2n^2 + 4n + 2 \\ &> (5n + 5) + 4n + 2 && \text{(by the induction hypothesis)} \\ &= 9n + 7 \\ &= \big(5(n+1) + 5\big) + (4n - 3). \end{aligned}$$

Now

$$n \geq 4 \implies 4n \geq 16 \implies 4n - 3 \geq 13 > 0.$$

Thus

$$2(n+1)^2 > \big(5(n+1) + 5\big) + (4n - 3) > 5(n+1) + 5,$$

completing the proof of the induction step.

QUESTION 6. **[4 points]** Use modular arithmetic to show that, for all $n \in \mathbb{Z}$,

$$n^5 + 8n^4 + 9n^3 - 26n^2 - 52n - 24$$

is divisible by 3.

Solution: We work modulo 3 since then $3 \mid k$ if and only if $[k] = [0]$. We have

$$\begin{aligned}
&[n^5{+}8n^4 + 9n^3 - 26n^2 - 52n - 24] \\
&\quad = [n^5] \oplus [8] \otimes [n^4] \oplus [9] \otimes [n^3] \oplus [-26] \otimes [n^2] \oplus [-52] \otimes [n] \oplus [-24] \\
&\quad = [n^5] \oplus [2] \otimes [n^4] \oplus [0] \otimes [n^3] \oplus [1] \otimes [n^2] \oplus [2] \otimes [n] \oplus [0] \\
&\quad = [n]^5 \oplus [2] \otimes [n]^4 \oplus [n]^2 \oplus [2n]
\end{aligned}$$

We now only need to check the three possibilities for $[n]$: $[0]$, $[1]$, $[2] = [-1]$. For $[n] = [0]$, we have

$$[0]^5 \oplus [2] \otimes [0]^4 \oplus [0]^2 \oplus [0] = [0].$$

For $[n] = [1]$, we have

$$[1]^5 \oplus [2] \otimes [1]^4 \oplus [1]^2 \oplus [2] = [1] \oplus [2] \oplus [1] \oplus [2] = [6] = [0].$$

Finally, for $[n] = [-1]$, we have

$$[-1]^5 \oplus [2] \otimes [-1]^4 \oplus [-1]^2 \oplus [-2] = [-1] \oplus [2] \oplus [1] \oplus [-2] = [0].$$

This completes the proof.

Alternate solution: By Fermat's Little Theorem, $n^3 \equiv n \mod 3$. Thus, starting as above, we have

$$\begin{aligned}
n^5 + 8n^4 + 9n^3 - 26n^2 - 52n - 24 &\equiv n^5 + 2n^4 + n^2 + 2n \\
&\equiv n^2 \cdot n^3 + 2n \cdot n^3 + n^2 + 2n \\
&\equiv n^2 \cdot n + 2n \cdot n + n^2 + 2n \\
&\equiv n^3 + 3n^2 + 2n \\
&\equiv n + 2n \\
&\equiv 0.
\end{aligned}$$

QUESTION 7. [**5 points**] Let $\sim$ be the relation on $\mathbb{R}_{>0}$ defined by

$$r \sim s \iff rs^{-1} \in \mathbb{Q}.$$

Prove that $\sim$ is an equivalence relation.

Solution: *Reflexivity*: Suppose $r \in \mathbb{R}_{>0}$. Then

$$rr^{-1} = 1 \in \mathbb{Q}.$$

So $r \sim r$.

Symmetry: Suppose $r, s \in \mathbb{R}_{>0}$ satisfy $r \sim s$. Thus $rs^{-1} \in \mathbb{Q}$. Since $r, s \neq 0$, we have $rs^{-1} \neq 0$. So $rs^{-1} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $a, b \neq 0$. Then

$$sr^{-1} = \left(rs^{-1}\right)^{-1} = \frac{b}{a} \in \mathbb{Q}.$$

So $s \sim r$.

Transitivity: Suppose $r, s, t \in \mathbb{R}_{>0}$ satisfy $r \sim s$ and $s \sim t$. So

$$rs^{-1} = \frac{a}{b} \qquad \text{and} \qquad st^{-1} = \frac{c}{d}$$

for some $a, b, c, d, \in \mathbb{Q}$, $b, d \neq 0$. Then

$$rt^{-1} = \left(rs^{-1}\right)\left(st^{-1}\right) = \frac{a}{b}\frac{c}{d} = \frac{ab}{bd} \in \mathbb{Q}.$$

So $r \sim t$.

QUESTION 8. [**7 points**] Consider the set

$$A = \left\{ \frac{2}{n} - 6 : n \in \mathbb{N} \right\}.$$

(a) Find the largest element (i.e. maximum) of A or prove that it does not exist.

Solution: We will show that $\max A = -4$. First note that, when $n = 1$, we have

$$\frac{2}{1} - 6 = -4.$$

So $-4 \in A$. Since

$$\begin{aligned} n \in \mathbb{N} &\implies n \geq 1 \\ &\implies \frac{1}{n} \leq 1 \\ &\implies \frac{2}{n} \leq 2 \\ &\implies \frac{2}{n} - 6 \leq -4, \end{aligned}$$

we also see that -4 is an upper bound of A. Thus $\max A = -4$.

(b) Find the supremum of A or prove that it does not exist.

Solution: Since A has a largest element, it has a supremum and

$$\sup A = \max A = -4.$$

(c) Find the infimum of A or prove that it does not exist.

Solution: We will show that $\inf A = -6$. First note that

$$\begin{aligned} n \in \mathbb{N} &\implies n > 0 \\ &\implies \frac{1}{n} > 0 \\ &\implies \frac{2}{n} > 0 \\ &\implies \frac{2}{n} - 6 > -6. \end{aligned}$$

So -6 is a lower bound for A. It remains to show that it is the *greatest* lower bound, which we do by contradiction. Suppose y is a lower bound for A with $y > -6$ (so $y + 6 > 0$). We will arrive at a contradiction by finding $n \in \mathbb{N}$ such that $\frac{2}{n} - 6 < y$. Note that

$$\frac{2}{n} - 6 < y \iff \frac{2}{n} < y + 6 \iff \frac{2}{y+6} < n.$$

Since $\mathbb{N}$ is not bounded above, there exists $n \in \mathbb{N}$ such that $\frac{2}{y+6} < n$. By the above, this n satisfies $\frac{2}{n} - 6 < y$, and so y is not a lower bound for A.

(d) Find the smallest element (i.e. minimum) of A or prove that it does not exist.

Solution: If A had a smallest element, we would have $\min A = \inf A = -6$. However, as shown above, all elements of A are *strictly* greater than -6. So $-6 \notin A$. Hence A does not have a smallest element.

QUESTION 9. [**6 points**] Consider the function

$$f\colon \mathbb{R} \to \mathbb{R}_{\geq 0}, \quad f(x) = (x-1)^2.$$

(a) Is f injective? Justify your answer.

Solution: No, f is not injective. For example,

$$f(2) = (2-1)^2 = 1 = (0-1)^2 = f(0).$$

(b) Is f surjective? Justify your answer.

Solution: Yes. Suppose $y \in \mathbb{R}_{\geq 0}$. Then $y \geq 0$, and so $\sqrt{y}$ is defined. Let $x = \sqrt{y} + 1$. Then

$$f(x) = \big((\sqrt{y}+1) - 1\big)^2 = (\sqrt{y})^2 = y,$$

since $y \geq 0$. So f is surjective.

(c) Does f have a left inverse? If it does, give one and show that it is indeed a left inverse. Otherwise, justify why f does not have a left inverse.

Solution: No. A function has a left inverse if and only if it is injective. Since f is not injective, it does not have a left inverse.

(d) Does f have a right inverse? If it does, give one and show that it is indeed a right inverse. Otherwise, justify why f does not have a right inverse.

Solution: Yes. A function has a right inverse if and only if it is surjective. Since f is surjective, it has a right inverse. Define

$$g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad g(y) = \sqrt{y} + 1.$$

Then, as above, for all $y \in \mathbb{R}_{\geq 0}$, we have

$$f(g(y)) = \big((\sqrt{y}+1) - 1\big)^2 = (\sqrt{y})^2 = y,$$

since $y \geq 0$. Thus $f \circ g = \mathrm{id}_{\mathbb{R}_{\geq 0}}$, and so g is a right inverse to f.

QUESTION 10. [**4 points**] Find the limit of the sequence

$$\left(\frac{5}{2}+\frac{1}{n^2+1}\right)_{n=1}^{\infty}.$$

Prove your answer directly using the definition of a limit. That is, do not use any results we proved about limits of particular sequences or the arithmetic of limits.

Solution: We will show that

$$\lim_{n\to\infty}\left(\frac{5}{2}+\frac{1}{n^2+1}\right)=\frac{5}{2}.$$

Let $\varepsilon > 0$. Note that

$$\begin{aligned}\left|\frac{5}{2}+\frac{1}{n^2+1}-\frac{5}{2}\right| &= \left|\frac{1}{n^2+1}\right| \\ &= \frac{1}{n^2+1} && \text{(since } n \geq 1 \text{ implies } n^2+1>0) \\ &< \frac{1}{n^2} && \text{(since } n^2+1>n^2>0) \\ &\leq \frac{1}{n}. && \text{(since } n\geq 1 \text{ implies } n^2 \geq n)\end{aligned}$$

Since $\mathbb{N}$ is unbounded, we can choose $N \in \mathbb{N}$ such that $N \geq \frac{1}{\varepsilon}$. Then, for all $n \geq N$, we have

$$\left|\frac{5}{2}+\frac{1}{n^2+1}-\frac{5}{2}\right| < \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon.$$