University of Ottawa

Department of Mathematics and Statistics

MAT 1362: Mathematical Reasoning & Proofs

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Midterm Test – Solutions

2 November 2017

Surname ________________________ First Name ______________________

Student # ______________________ DGD (1 or 2) ______________________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this *clearly*. Otherwise, the work written on the reverse side of pages will not be considered for marks.
(e) Write your student number at the top of each page in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You should write in *pen*, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	Total
Maximum	2	4	2	2	6	3	19
Grade							

Question 1. [**2 points**]

(a) Give the definition of subtraction of integers.

Solution: If $a, b \in \mathbb{Z}$, then $a - b = a + (-b)$, where $-b$ is the additive inverse of b.

(b) Prove that if $a, b, c \in \mathbb{Z}$ with $a < b$, then $a - c < b - c$. You may only use the properties of the integers we have learned (basic arithmetic of integers), the defining properties of the natural numbers (closure under addition, closure under multiplication, etc.) and the definition of $<$. Do *not* use any propositions we have proved about the order $<$ on the integers. You do *not* need to show all steps related to distributivity, associativity, etc.

Solution: Suppose $a, b, c \in \mathbb{Z}$ with $a < b$. Then, by the definition of $<$, we have $b - a \in \mathbb{N}$. Thus

$$(b - c) - (a - c) = b - c - a - (-c) = b - a \in \mathbb{N}.$$

Thus $a - c < b - c$.

QUESTION 2. [**4 points**] Prove by induction that

$$\sum_{i=1}^{k}(3i-2) = \frac{k(3k-1)}{2} \quad \text{for all } k \in \mathbb{N}.$$

Solution: *Base case*: When $k = 1$, we have

$$\sum_{i=1}^{1}(3i-2) = 3 \cdot 1 - 2 = 1 = \frac{1(3 \cdot 1 - 1)}{2}.$$

Thus the result holds for $k = 1$.

Induction step: Suppose that, for some $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n}(3i-2) = \frac{n(3n-1)}{2}.$$

Then

$$\begin{aligned}\sum_{i=1}^{n+1}(3i-2) &= \left(\sum_{i=1}^{n}(3i-2)\right) + \big(3(n+1)-2\big) \\ &= \frac{n(3n-1)}{2} + 3n + 1 \\ &= \frac{3n^2 - n + 6n + 2}{2} \\ &= \frac{3n^2 + 5n + 2}{2} \\ &= \frac{(n+1)(3n+2)}{2} \\ &= \frac{(n+1)(3(n+1)-1)}{2}.\end{aligned}$$

This completes the proof of the induction step.

QUESTION 3. [**2 points**] For each of the following statements, write 'T' if the statement is true and write 'F' if the statement is false. You do not need to justify your answers.

Grading: You will receive 0.5 points for each correct answer, lose 0.25 points for each incorrect answer, and receive zero points for an answer left blank. Your grade on this question will then be rounded down to the nearest 0.5 points. You cannot receive a negative score on this question.

T $(0 = 1) \implies (3 > 2)$.

T $(\forall m \in \mathbb{Z})\,(\exists n \in \mathbb{Z}$ such that$)\, m + n = 1$.

F $(\exists a \in \mathbb{Z}$ such that$)\,(\forall b \in \mathbb{Z})\, ab \neq 0$.

F $(\exists p \in \mathbb{Z}$ such that$)(\forall q \in \mathbb{N})\, q < p$.

QUESTION 4. [**2 points**] Find the negations of the following statements. Simplify your answers as much as possible.

(a) $(a = bc) \implies (b \leq d)$.

Solution: $(a = bc)$ and $(b > d)$.

(b) $(m = n^2)$ or every cow is pink.

Solution: $(m \neq n^2)$ and there exists a cow that is not pink.

(c) $(\exists a \in \mathbb{Z}$ such that$)\,(\forall b \in \mathbb{N})\,(a \leq b$ and $ab = 1)$.

Solution: $(\forall a \in \mathbb{Z})\,(\exists b \in \mathbb{N}$ such that$)\,(a > b$ or $ab \neq 1)$.

(d) $(\forall x \in \mathbb{Z})\,(\exists y \in \mathbb{Z}_{\geq 0}$ such that$)\, xy + 7x^2 \neq 3$.

Solution: $(\exists x \in \mathbb{Z}$ such that$)\,(\forall y \in \mathbb{Z}_{\geq 0})\, xy + 7x^2 = 3$.

QUESTION 5.

(a) [**3 points**] Let

$$A = \{4a + 1 : a \in \mathbb{Z}\} \quad \text{and} \quad B = \{4b - 7 : b \in \mathbb{Z}\}.$$

Prove that $A = B$.

Solution: Suppose $x \in A$. Then there exists $a \in \mathbb{Z}$ such that $x = 4a + 1$. Then

$$x = 4a + 1 = 4a + 8 - 7 = 4(a + 2) - 7 = 4b - 7,$$

where $b = a + 2 \in \mathbb{Z}$. Thus $x \in B$. So $B \subseteq A$.

Now suppose $y \in B$. Then there exists $b \in \mathbb{Z}$ such that $y = 4b - 7$. Then

$$y = 4b - 7 = 4b - 8 + 1 = 4(b - 2) + 1 = 4a + 1,$$

where $a = b - 2 \in \mathbb{Z}$. Thus $y \in A$. So $A \subseteq B$. Hence $A = B$.

(b) [**3 points**] Suppose X and Y are sets. Prove that

$$X - (X \Delta Y) = X \cap Y.$$

Solution: We have

$$\begin{aligned} x \in X - (X\Delta Y) &\iff x \in X \text{ and } x \notin X\Delta Y \\ &\iff x \in X \text{ and } \neg\big(x \in (X - Y) \cup (Y - X)\big) \\ &\iff x \in X \text{ and } \neg\big(x \in X - Y \text{ or } x \in Y - X\big) \\ &\iff x \in X \text{ and } x \notin X - Y \text{ and } x \notin Y - X \\ &\iff x \in X \text{ and } x \in Y \\ &\iff x \in X \cap Y, \end{aligned}$$

where the second-to-last "$\iff$" follows from the fact that if $x \in X$, then $x \notin Y - X$ and

$$x \notin X - Y \iff x \in Y.$$

QUESTION 6. [**3 points**] Define a relation $\sim$ on the set $\mathbb{N} \times \mathbb{N}$ by

$$(a, b) \sim (c, d) \iff ad = bc.$$

Prove that $\sim$ is an equivalence relation.

Solution: *Reflexivity*: Suppose $(a, b) \in \mathbb{N} \times \mathbb{N}$. Since $ab = ba$, we have $(a, b) \sim (a, b)$. So $\sim$ is reflexive.

Symmetry: Suppose $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$ such that $(a, b) \sim (c, d)$. Then $ad = bc$. Thus $cb = da$, and so $(c, d) \sim (a, b)$. Hence $\sim$ is symmetric.

Transitivity: Suppose $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $ad = bc$ and $cf = de$. So

$$adf = bcf = bde.$$

Since $d \neq 0$, the cancellation property gives $af = be$. Hence $(a, b) \sim (e, f)$. So $\sim$ is transitive.